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Some Properties of the Exponential Mean.

By Julia Dale.

1. Introduction.

Among the many definitions that have been given for evaluating a divergent series, the exponential mean is frequently one of the most convenient. In this paper, the purpose has been to study some of the properties of this definition and show its relation to Euler's Summation Process.

Unless otherwise stated, we use the symbol

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots,$$

and define

$$x_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

The exponential mean * assigns to a divergent series the value $\lim_{n\to\infty} y_n$, if this limit exists, where

$$(E_r) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} x_k.$$

The successive coefficients are the terms of the binomial expansion of

$$[(1-1/r) + 1/r]^{n-1}.$$

The transformation which we have designated in this paper as Euler's Summation Process was defined by Euler,† when applied to a series of decreasing terms of alternating signs, in the following manner:

$$S = s_{n-1} + (-1)^{n-1}[\tfrac{1}{2}u_n + \tfrac{1}{4}\Delta^1 u_n + \tfrac{1}{8}\Delta^2 u_n + \cdots + \tfrac{1}{2^k}\Delta^{k-1}u_n].$$

where n is an integer and

$$s_{n-1} = u_1 - u_2 + u_3 + \cdots + (-1)^{n-1}u_{n-1}$$

and

$$\Delta^1 u_n = u_n - u_{n+1},$$

$$\Delta^2 u_n = \Delta^1 u_n - \Delta^1 u_{n+1} = u_n - 2u_{n+1} + u_{n+2},$$

. . . . . .

* Hurwitz: "Report on topics in the theory of divergent series," *Bulletin of the American Mathematical Society*, Vol. 28 (1922), p. 17. This paper will be referred to as *Report*.

Also, Hausdorff: "Summationsmethoden und Momentfolgen," *Mathematische Zeitschrift*, Vol. 9 (1921), p. 86.

† Euler: *Institutiones Calculi differentialis*, Part II, De transformatione serierum, p. 232.

Let us use this method to evaluate the series *

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots .$$

The sum of the first eight terms to six decimals is 0.634524. Applying the transformation to the next seven terms, we obtain

$$\begin{aligned} S &= .634524 + \tfrac{1}{2}(.111111) + \tfrac{1}{4}(.011111) + \tfrac{1}{8}(.002020) + \\ &\quad 1/16(.000505) + 1/32(.000156) + 1/64(.000057) \\ &= .634524 + .058624 \\ &= .693148. \end{aligned}$$

The actual sum of the series is $\log 2 = .69314718$.

"To reach this degree of accuracy we should have to use over a hundred thousand terms of the original series." *

The first discussion of the accuracy of this transformation is due to Poncelet.† He applies this transformation to slowly convergent series of alternating signs and computes a formula for the limit of error, when we stop with the nth term. Poncelet, also, defines a generalized form of this transformation. It is designated in this paper as Euler's Generalized Summation Process. Kummer,‡ Markoff,§ and Catalan ‖ have worked with different forms of this transformation, principally with the original form. Ames ¶ has computed a formula for the limit of error, when the original form of Euler's Summation Process is applied to slowly convergent series of certain types. He has also suggested that, if this transformation be applied to certain types of divergent series, it renders them convergent.

In this paper we have applied this transformation in its original form and in the generalized form to divergent series and studied some of its properties. In parts 2 and 3 it is proved that not only the original Euler's Summation Process but also the generalized one is a regular ** definition.

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* Bromwich: *Theory of Infinite Series*, (1908), p. 58.

† Poncelet: "Application de la méthode des Moyennes à la transformation, au calcul numérique et à la détermination des limites du reste des séries," *Journal für Mathematik von Crelle*, Vol. 13 (1833), p. 1.

‡ Kummer: "Eine neue Methode, die Numerischen Summen langsam convergierender Reihen zu berchnen," *Journal für Mathematik von Crelle*, Vol. 16 (1837), p. 206.

§ Markoff: *Differenzenrechnung*, Leipzig, (1896).

‖ Catalan: "Mémoire sur la Transformation des Séries," *Mémoires Courronés de l'Académie de Belgique*, Vol. 33 (1867), p. 2.

¶ Ames: "Evaluation of Slowly Convergent Series," *Annals of Mathematics*, Series 2, Vol. 3 (1902), p. 185.

** The terminology here used is that of Professor Hurwitz, *Report*.

A definition is said to be *regular,* if it evaluates every convergent sequence, giving it the value to which it converges. Also, it is proved that this definition is equivalent * to the exponential mean. Two definitions are defined as being *equivalent* when each evaluates to the value l every sequence evaluated by the other to the value l.

Part 4 is devoted to the application of the exponential mean to the Cauchy product of two series. Knopp ‡ built a definition by applying E_2 successively p times to the given series. He investigated some of the properties of this definition, among them the value assigned, if one exists, to the Cauchy product of two series. He proved some theorems that are generalizations of theorems of Abel and Mertens for convergent series. Here we give an explicit definition of summability which evaluates the Cauchy product of two series summable respectively E_r and E_s.

2. The Euler Summation Process.

We consider the Euler process as applied directly from the first term, that is with $n = 1$ in the formula given in the Introduction.

For the series

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

define

1) $$s_n = \sum_{k=1}^{n} \tfrac{1}{2}^k \Delta^{k-1} u_1,$$

where

$$\Delta^0 u_1 = u_1,$$
$$\Delta^1 u_1 = u_1 - u_2,$$
$$\Delta^2 u_1 = \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\Delta^k u_1 = \Delta^{k-1} u_1 - \Delta^{k-1} u_2$$
$$= u_1 - k u_2 + \frac{k(k-1)}{1 \cdot 2} u_3 - \cdots + (-1)^k u_{k+1},$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad ;$$

then the generalized value of the series is to be

$$s = \lim_{n \to \infty} s_n.$$

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* The terminology here used is that of Professor Hurwitz, *Report.*

‡ Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift,* Vol. 15 (1922-23), p. 226; Vol. 18 (1923), p. 125.

In order that the definition may be applied to a series written in the usual way with positive signs, we define

$$v_n = (-1)^{n-1}u_n.$$

We have

$$s_n = \sum_{k=1}^{n} 1/2^k[u_1 - (k-1)u_2 + \frac{(k-1)(k-2)}{1\cdot 2}\, u_3 - \cdots + (-1)^{k-1}u_k],$$

or, by inversion of the order of summation,

$$2) \qquad s_n = \sum_{k=1}^{n}\sum_{h=1}^{n-k+1} \frac{(-1)^{k-1}}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}\, u_k.$$

Thus the relation of s_n to v_n is given by

$$(\mathcal{E}_2) \qquad s_n = \sum_{k=1}^{n} a_{n,k}v_k,$$

where

$$3) \qquad a_{n,k} = \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}.$$

THEOREM I. *$\mathcal{E}_2$ is a regular transformation.*

In order that a transformation be regular, when applied directly to the terms of the series, it is necessary and sufficient that *

a) for each k, $\lim_{n\to\infty} a_{n,k} = 1$

b) for all n $\sum_{k=1}^{n-1} |\, a_{n,k} - a_{n,k+1}\,|$ is bounded.

Obviously, condition a) is satisfied, since

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}$$

$$= \sum_{h=1}^{\infty} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}$$

$$= 1/2^k\,[1-\tfrac{1}{2}]^{-k} = 1.$$

In order to establish condition b), we write 3) in the form

$$a_{n,k} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}.$$

* Carmichael: "General Aspects of the Theory of Summable Series," *Bulletin of the American Mathematical Society*, Vol. 25 (1916-1919), p. 97.

Then

$$4) \quad a_{n,k} - a_{n,k+1} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k}\right]$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k} + \frac{2}{k}\right]$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k-1}} \frac{(h+k-1)!}{(h-1)!\,k!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$- \sum_{h=0}^{n-k-1} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$= \frac{1}{2^n} \frac{n!}{(n-k)!\,k!} .$$

From this, we see that

$$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} | = \frac{1}{2^n} [(1+1)^n - 2] = 1 - \frac{1}{2^{n-1}} < 1$$

Then

$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} |$ is bounded for all n and $\mathcal{E}_2$ satisfies the conditions for regularity.

We shall now express this transformation in the form of a sequence-to-sequence transformation. Form the sequence

$$5) \qquad x_1, x_2, x_3, x_4, \cdots$$

where $x_k = v_1 + v_2 + \cdots + v_k.$

then $v_k = x_k - x_{k-1}$

and

$$(\mathcal{E}_2) \qquad s_n = \sum_{k=1}^{n} a_{n,k}(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} a_{n,k} x_k - \sum_{k=0}^{n-1} a_{n,k+1} x_k$$

$$= \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1}) x_k + a_{n,n} x_n.$$

Thus

$$s_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = \begin{cases} a_{n,k} - a_{n,k+1} & k < n, \\ a_{n,n}, & k = n. \end{cases}$$

Furthermore, by 4), we find

$$b_{n,k} = \frac{1}{2^n} \cdot \frac{n!}{(n-k)!\,k!}$$

THEOREM II. *$\mathcal{E}_2$ is equivalent to the exponential mean E_2.*

In this work the notation $A \sim B$ will be used to indicate that two given sequences A and B are equivalent.

Define

(P) $$x'_k = \begin{cases} x_{k-1} & k > 1, \\ 0 & k = 1. \end{cases}$$

It is readily verified that

6) $$E_2 P = P \mathcal{E}_2.$$

The transformation $\mathcal{E}_2$ has been shown to be regular, therefore it is obvious that

7) $$\mathcal{E}_2 \sim P \mathcal{E}_2.$$

If a sequence is summable E_r, then the sequence obtained by prefixing an element is summable to the same value,* therefore

8) $$E_2 \sim E_2 P.$$

By means of 6), 7), 8), we see that

$$E_2 \sim \mathcal{E}_2.$$

3. THE GENERALIZED EULER SUMMATION PROCESS.

For the generalized Euler Summation Process, as applied to the series

$$u_1 t - u_2 t^2 + u_3 t^3 - u_4 t^4 + \cdots,$$

we define

9) $$s_n = \sum_{k=1}^{n} (t/1+t)^k \Delta^{k-1} u_1,$$

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* Hurwitz, *Report.*

where

$$\Delta^0 u_1 = u_1,$$
$$\Delta^1 u_1 = u_1 - u_2,$$
$$\Delta^2 u_1 = \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Delta^k u_1 = \Delta^{k-1} u_1 - \Delta^{k-1} u_2 = u_1 - ku_2 + \frac{k(k-1)}{1 \cdot 2} u_3 - \cdots$$
$$\cdots + (-1)^k u_{k+1}$$

then the generalized value of the series is to be given by

$$s = \lim_{n \to \infty} s_n.$$

In order to find an expression for s_n, involving directly the terms of the series, we define

$$v_n = (-1)^{n-1} u_n t^n.$$

We have

$$s_n = \sum_{k=1}^{n} (t/1+t)^k \left[u - \frac{(k-1)}{1!} u_2 + \frac{(k-1)(k-2)}{1 \cdot 2} u_3 \cdots + (-1)^{k-1} u_k\right].$$

or, by inversion of the order of summation,

10) $$s_n = \sum_{k=1}^{n} \sum_{h=1}^{n-k+1} \frac{(-1)^{k+1}(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}} u_k t^k.$$

Thus the relation of s_n to v_n is given by

$$(\mathcal{E}_{1+t}) \quad s_n = \sum_{k=1}^{n} a_{n,k} v_k,$$

where

11) $$a_{n,k} = \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}.$$

THEOREM III. *$\mathcal{E}_{1+t}$, where $t > 0$, is a regular transformation.*

Returning to the conditions for regularity as stated in part 2, we find condition a) satisfied, since

$$\lim_{n \to \infty} a_{n,k} = \lim_{n \to \infty} \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \sum_{h=1}^{\infty} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \frac{1}{(1+t)^k} \left[1 - \frac{t}{1+t}\right]^{-k} = 1.$$

In order to establish condition b) we write

$$12) \qquad a_{n,k} - a_{n,k+1} = \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k}}.$$

Changing the variable of summation in the first sum, we have

$$a_{n,k} - a_{n,k+1} = \sum_{h=0}^{n-k} \frac{(h+k-1)!}{h!(k-1)!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k}}$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \left[k - \frac{h}{t}\right]$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \left[k - \frac{h}{t} + \frac{h}{t} + h\right] - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \frac{h(1+t)}{t}$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k-1}}.$$

Changing the variable of summation in the second sum, we have

$$a_{n,k} - a_{n,k+1} = \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=0}^{n-k-1} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}}$$

$$= \frac{t^{n-k}}{(1+t)^n} \frac{n!}{(n-k)!\,k!}$$

Hence, we see that

$$\sum_{k=1}^{n-1} |\, a_{n,k} - a_{n,k+1} \,| = (t/1+t)^n \left[(1+1/t)^n - 1 - 1/t^n\right]$$

$$= 1 - (t/1+t)^n - 1/(1+t)^n < 1.$$

Consequently, condition b) is satisfied.*

Let us express the generalized Euler summation transformation in the

---

* We may choose to consider $\mathcal{E}_{1+t}$ with coefficients given by 20) for any value of t, with the convention that $\mathcal{E}_1$ is the identity. The method of the text can readily be extended to show that $\mathcal{E}_{1+t}$ is regular if and only if t is real and ≥ 0. Furthermore, for these values $\mathcal{E}_{1+t}$ is totally regular, in the sense of Professor Hurwitz's *Report*, p. 29.

form of a sequence-to-sequence transformation. As in the preceding case, we find,

$$s_n = \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1}) x_k + a_{n,n} x_n,$$

Thus

$$s_n = \sum_{k=1}^{n} \beta_{n,k} x_k,$$

where

$$\beta_{n,k} = \frac{t^{n-k}}{(1+t)^n} \quad \frac{n!}{(n-k)!\,k!}$$

THEOREM IV. *$\mathcal{E}_{1+t}$, for $t > 0$, is equivalent to the exponential mean E_{1+t}.*

As before we can show that

13) $$E_{1+t} P = P\, \mathcal{E}_{1+t}.$$

The transformation $\mathcal{E}_{1+t}$ has been shown to be regular, then it is obvious that

14) $$\mathcal{E}_{1+t} \sim P\, \mathcal{E}_{1+t}$$

as also *

15) $$E_{1+t} \sim E_{1+t} P.$$

It follows from 13), 14), 15) that

$$E_{1+t} \sim \mathcal{E}_{1+t}.$$

THEOREM V. *$\mathcal{E}_{1+t}\ \mathcal{E}_{1+s} = \mathcal{E}_{(1+t)(1+s)}$, for $t > 0$ and $s > 0$.*

Let us apply $\mathcal{E}_{(1+t)(1+s)}$ to 5), then we have

16) $$w_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \quad \frac{(t+s+ts)^{n-k}}{(1+t)^n (1+s)^n}\ x_k.$$

Applying $\mathcal{E}_{1+t}$ to sequence 5) and then $\mathcal{E}_{1+s}$, we find

$$y_n = \sum_{h=1}^{n} \frac{n!}{(n-h)!\,h} \quad \frac{t^{n-h}}{(1+t)^n} \sum_{k=1}^{h} \frac{h!}{(h-k)!\,k!} \quad \frac{s^{h-k}}{(1+s)^h}\ x_k.$$

$$= \sum_{k=1}^{n} (t/1+t)^n x_k \sum_{h=k}^{n} \frac{n!}{(n-h)!\,(h-k)!\,k!} \quad \frac{1}{t^h} \quad \frac{s^{h-k}}{(1+s)^h}$$

$$= \sum_{k=1}^{n} (t/1+t)^n \frac{n!}{(n-k)!\,k!} \quad \frac{1}{t^k} \quad \frac{1}{(1+s)^k} \left[1 + \frac{s}{t(1+s)}\right]^{n-k} x_k.$$

* Hurwitz: *Report*, p. 31.

that is

17) $$y_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \frac{(t+s+ts)^{n-k}}{(1+t)^n(1+s)^n} x_k.$$

Therefore from 16) and 17) we have

$$w_n = y_n.$$

By a successive application of this result, we infer

COROLLARY 1. $\mathcal{E}_{(1+t_1)(1+t_2)\cdots(1+t_n)} = \mathcal{E}_{1+t_1} \mathcal{E}_{1+t_2} \cdots \mathcal{E}_{1+t_n}$.

COROLLARY 2. $\mathcal{E}^p_{1+t} = \mathcal{E}_{(1+t)^p}$.

THEOREM VI. *If a sequence is summable $\mathcal{E}_{1+t}$ to the value l, then any sequence obtained by omitting or prefixing a finite number of terms is summable $\mathcal{E}_{1+t}$ to the value l.*

If a sequence is summable E_{1+t} to the value l, then the sequence obtained by omitting or prefixing a term is summable E_{1+t} to the value l.* It was proved Theorem IV, that

$$E_{1+t} \sim \mathcal{E}_{1+t}$$

By means of the definition of equivalence $\mathcal{E}_{1+t}$ enjoys the same property.

It is sometimes convenient to sum by ordinary addition the first p terms of the series

$$u_1 t - u_2 t^2 + u_3 t^3 + \cdots$$

and then apply the generalized Euler Summation Process to the remaining terms. The generalized sum of the series, if it exists, would then be defined as

$$\lim_{n\to\infty} s_n$$

where

$$s_n = s_p + \sum_{k=1}^{n} (t/1+t)^k \Delta^{k-1} u_{p+1},$$

and

$$s_p = u_1 t - u_2 t^2 + \cdots + (-1)^{p-1} u_p t^p.$$

But here again, in order to find an expression for s_n, involving directly the terms of the series, we make the linear transformation

$$v_n = (-1)^{n-1} u_n t^n.$$

Then we have

$(\mathcal{E}_{1+t,p})$ $$s_n = x_p + \sum_{k=1}^{n} a_{n,k} v_{p+k}$$

where $a_{n,k}$ is given by 11).

* Hurwitz: *Report*, p. 31.

THEOREM VII. *$\mathcal{E}_{1+t,p}$ is equivalent to $\mathcal{E}_{1+t}$.*

We can obviously write

$$\mathcal{E}_{1+t,p}(x_n) = x_p + \mathcal{E}_{1+t}(x_{p+n} - x_p)$$
$$= [x_p - \mathcal{E}_{1+t}(x_p)] + \mathcal{E}_{1+t}(x_{p+n}).$$

The expression in the brackets has the limit zero as $n \to \infty$ since $\mathcal{E}_{1+t}$ is regular. Thus if $\mathcal{E}_{1+t,p}$ evaluates (x_n) to l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l, and by Theorem VI, $\mathcal{E}_{1+t}$ will evaluate (x_n) to l. Conversely, if $\mathcal{E}_{1+t}$ evaluates (x_n) to l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l and $\mathcal{E}_{1+t,p}$ will evaluate (x_n) to l. Hence we may state

$$\mathcal{E}_{1+t,p} \sim \mathcal{E}_{1+t}.$$

4. MULTIPLICATION OF SERIES.

The expressions obtained by multiplying the terms of one series $\sum u_n$ by the terms of another series $\sum u_n$ may be arranged formally as a double series $\sum u_m v_n$. We may collect the terms of this double series into a simple series $\sum w_n$, by means of the Cauchy product, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

Before taking up the case of the summability of the Cauchy product of two series, which are summable E_r, we prove two lemmas.

Take two sequences

$$(x_n) : x_1, x_2, x_3, \cdots,$$
$$(y_n) : y_1, y_2, y_3, \cdots,$$

and an infinite matrix

$$(a_{n,k}) : \left\| \begin{array}{ccccccccc} a_{11}, & 0, & 0, & 0, & . & . & . & . & . \\ a_{21}, & a_{22}, & 0, & 0, & . & . & . & . & . \\ a_{31}, & a_{32}, & a_{33}, & 0, & 0, & . & . & . & . \\ . & . & . & & & & & & \\ . & . & . & & & & & & \\ . & . & . & & & & & & \\ . & . & . & & & & & & \end{array} \right\| .$$

Then the transformation

18) $$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

defines the sequence

$$(z_n) : z_1, z_2, z_3, \cdots .$$

We indicate the transformation 18) by $\mathcal{A}$, and the relation by

$$z_n = \mathcal{A}(x_n y_n).$$

LEMMA 1. *The necessary and sufficient conditions that, if* $\lim_{n\to\infty} y_n = y$ *and* $\lim_{n\to\infty} x_n = x$, *then* $\lim_{n\to\infty} z_n = xy$, *where*

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

are

a) $\lim_{n\to\infty} a_{n,k} = 0$ *for each* k;

b) $\lim_{n\to\infty} a_{n,n-k+1} = 0,$ *for each* k;

c) $\sum_{k=1}^{n} | a_{n,k} | < c,$ *for all* n;

d) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1.$

We shall first establish the necessity of these conditions.

Take

$$y_1 = y_2 = y_n = \cdots = 1$$

and leave (x_n) arbitrary. Substiuting this value of (y_n) in 18), we have

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k.$$

This is now an ordinary sequence-to-sequence transformation. For a transformation of this type to be regular, it is necessary and sufficient that *

a′) $\lim_{n\to\infty} a_{n,k} = 0,$ for each k

b′) $\sum_{k=1}^{n} | a_{n,k} | < c,$ for all n,

c′) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1.$

Hence, we have established the necessity of three of these conditions.

Take

$$x_1 = x_2 = x_3 = \cdots = 1$$

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* Hurwitz: *Report*, p. 19.

and keep (y_n) arbitrary. Substituting in 18) we have

$$z_n = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k.$$

This is a regular transformation; hence in addition to b′) and c′)

$$\lim_{n\to\infty} a_{n,n-k+1} = 0, \qquad \text{for each } k.$$

It remains to establish the sufficiency of these conditions. Write

19) $$z_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = a_{n,k} y_{n-k+1}$$

From condition a) we see that

20) $$\lim_{n\to\infty} b_{n,k} = 0, \qquad \text{for a fixed } k.$$

By condition c) and the fact that, for all n, $|\, y_n \,| \leq Y$, we have

21) $$\sum_{k=1}^{n} |\, b_{n,k} \,| = \sum_{k=1}^{n} |\, a_{n,k} \,|\, |\, y_{n-k+1} \,| < Y \cdot c.$$

We now write

$$\sum_{k=1}^{n} b_{n,k} = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k$$

Since the transformation having the matrix $(a_{n,n-k+1})$ is a regular transformation, we have

22) $$\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} = y$$

Choose p, such that given $\epsilon > 0$,

$$|\, x_k - x \,| < \epsilon/2Yc, \qquad k > p.$$

From 19) we may write

$$z_n = \sum_{k=1}^{p} b_{n,k} x_k + \sum_{k=p+1}^{n} b_{n,k} x_k,$$

23) $$|\, z_n - x \sum_{k=1}^{n} b_{n,k} \,| \leq \sum_{k=1}^{p} |\, b_{n,k} \,|\, |\, x_k - x \,| + \sum_{k=p+1}^{n} |\, b_{n,k} \,|\, |\, x_k - x \,|$$

For the fixed number p

$$| x_k - x | < M, \qquad k \leq p$$

Substituting these results in 23), we have

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq M \sum_{k=1}^{p} | b_{n,k} | + \epsilon/2Yc \sum_{k=p+1}^{n} b_{n,k}.$$

By 20), given $\epsilon > 0$, we may choose $N > p$, so that

$$| b_{n,k} | < \epsilon/2pM, \qquad n > N.$$

Thus we have, for $n > N$,

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

By 22) we see that

$$\lim_{n\to\infty} z_n = xy.$$

The matrix $\mathcal{A}$, which we have used to give a bilinear transformation 18) of two sequences, may be considered also as giving in the usual way a transformation of a single sequence

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k;$$

we may then speak of the algebraic properties of $\mathcal{A}$ with their usual meaning for transformations of this form.

A transformation of type $\mathcal{A}$ is pemutable with the fisrt Hölder mean, M, provided the matrix of $\mathcal{A}$ is permutable with the matrix of M.

LEMMA 2. *A necessary and sufficient condition that*

$$\mathcal{A}[E_r(x_n), E_r(y_n)] = E_r \mathcal{A}(xy).$$

is that the transformation $\mathcal{A}$ be permutable with M.

To show that this condition is necessary, we take the sequence

$$y_1 = y_2 = y_3 = \cdots = 1.$$

thus we have

$$24) \qquad \mathcal{A}[E_r(x_n), 1] = E_r \mathcal{A}(x_n, 1).$$

Define

$$25) \qquad \mathcal{A}(x_n, 1) = A(x_n),$$

where $A(x_n)$ is an ordinary sequence-to-sequence transformation.

Write Euler's Transformation *

$$26) \qquad y_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}(n-1)!}{(n-k)!(k-1)!} \; x_k,$$

which we shall call Δ. It satisfies the condition

$$\Delta^2 = I,$$

where I is the identity, as is evident by an actual repetition of 26).

Form the transformation $\Delta E_r \Delta$ and designate it by E'_r, similarly $M' = \Delta M \Delta$. The transformation E_r is permutable † with M, consequently

$$E'_r M' = M' E_r'$$

A necessary and sufficient condition that a transformation be permutable with M' is that it be a multiplication, i. e. of the form ‡

$$y_n = f_n x_n.$$

Hence, E'_r is a multiplication and may be expressed in the form

$$y_n = \frac{1}{r^{n-1}} \; x_n.$$

Form the transformation $A' = \Delta A \Delta$, and denote the elements of the matrix of A' by $(a'_{n,k})$. Applying $A'E'_r$ to (x_n), we have

$$27) \qquad \omega_n = \sum_{k=1}^{n} a'_{n,k} \; \frac{1}{r^{k-1}} \; x_k,$$

applying $E'_r A'$ to (x_n), we obtain

$$28) \qquad \Omega_n = \sum_{k=1}^{n} \frac{1}{r^{n-1}} \; a'_{n,k} x_k.$$

In order that $\omega_n = \Omega_n$, for all (x), we see from 27) and 28) that $a'_{n,k} = 0$, when $k \neq n$.

Hence, A' is a multiplication and we have at once that A is permutable with M. From 25) we see that the matrix of $\mathcal{A}$ is the same as the matrix of A, then $\mathcal{A}$ is permutable with M.

It now remains to prove the sufficiency of this condition. All trans-

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* Bromwich: *An Introduction to the Theory of Infinite Series* (1908), p. 303.

† Hurwitz and Silverman: "On the Consistency and Equivalence of Certain Definitions of Summability," *Transactions of American Mathematical Society*, Vol. 18, p. 6. This paper will be referred to as Hurwitz and Silverman.

‡ Hurwitz and Silverman, p. 6.

formations permutable with M may be expressed in the following form *

$$y_n = \sum_{k=1}^{n} \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h x_h.$$

$\mathcal{A}$ is in this class, so we may write

$$a_{n,k} = \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h.$$

In 18) replace (x_n) by $E_r(x_n)$ and (y_n) by $E_r(y_n)$, then if we write

$$Z_n = \mathcal{A}[E_r(x_n), E_r(y_n)]$$

we have

$$Z_n = \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h \sum_{l=1}^{k} \frac{(k-1)!(r-1)^{k-l}}{(k-l)!(l-1)!\, r^{k-1}} x_l \times \sum_{q=1}^{h-k+1} \frac{(h-k)!}{(h-k-q+1)!(q-1)!} \frac{(r-1)^{h-k-q+1}}{r^{h-k}} y_q,$$

$$29) \quad Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \sum_{q=1}^{h-l+1} \sum_{k=l}^{h-q+1} \frac{(n-1)!(-1)^{h-k} f_h}{(n-h)!(k-l)!(l-1)!(h-k-q+1)!(q-1)!} \frac{(r-1)^{h-l-q+1}}{r^{h-1}} x_l x_q.$$

We find

$$\sum_{k=l}^{h-q+1} \frac{(-1)^k}{(k-l)!(h-k-q+1)!} = \begin{cases} 0, & h+1 \neq q+l, \\ (-1)^l & h+1 = q+l. \end{cases}$$

Substituting this result in 29), we obtain

$$30) \quad Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \frac{(-1)^{h-l}(n-1)!}{(n-h)!(l-1)!(h-l)!} f_h \frac{1}{r^{h-1}} x_l y_{h-l+1}.$$

Applying E_r to (z_n) and writing

$$W_n = E_r \mathcal{A}(x_n, y_n),$$

we have

$$W_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \times \sum_{h=1}^{k} \sum_{l=1}^{h} \frac{(k-1)!(-1)^{h-l}}{(k-h)!(h-l)!(l-1)!} f_h x_l y_{h-l+1},$$

$$31) \quad W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \sum_{k=h}^{n} \frac{(n-1)!(-1)^{h-l} f_h}{(n-k)!(k-h)!(h-l)!(l-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} x_l y_{h-l+1}$$

we find that

$$\sum_{k=h}^{n} \frac{1}{(n-k)!(k-h)!} \frac{1}{(r-1)^k} = \frac{1}{(n-h)!} \frac{1}{(r-1)^h} \left[1 + \frac{1}{r-1}\right]^{n-h} = \frac{1}{(n-h)!} \frac{r^{n-h}}{(r-1)^n}$$

* Hurwitz and Silverman, p. 7.

When we substitute this result in 31), we see that

$$32) \quad W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \frac{(n-1)!(-1)^{h-l}}{(n-h)!(h-l)!(l-1)!} f_h \frac{1}{r^{h-1}} x_l y_{h-l+1};$$

By comparing results of 30) and 32), we find that

$$W_n = Z_n,$$

or

$$\mathcal{A}[E_r(x_n), E_r(y_n)] = E_r \mathcal{A}(x_n y_n)$$

as we wished to prove.

Take the two sequences (x_n) and (y_n) and the infinite matrix

$$(m_{nk}) : \left\| \begin{array}{ccccccccccc} 1/1, & 0, & 0, & 0, & . & . & . & . & . & . \\ 1/2, & 1/2, & 0, & 0, & . & . & . & . & . & . \\ 1/3, & 1/3, & 1/3, & 0, & 0, & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1/n, & 1/n, & 1/n, & . & . , & 1/n, & 0 & . & . & \\ . & . & . & . & . & . & . & . & . & . \end{array} \right\|$$

Then the transformation

$$33) \quad z_n = \sum_{k=1}^{n} m_{nk} x_k y_{n-k+1}$$

defines a sequence (z_n). We indicate the transformation by $\mathcal{M}$, and the relation by

$$(z_n) = \mathcal{M}(x_n y_n).$$

THEOREM VIII. *If $\sum u_n$ is summable E_r to the value σ and $\sum u_n$ is summable E_r to the value τ, then the Cauchy product $\sum w_n$ is summable ME_r to the value $\sigma\tau$.*

The two sequences formed from the two series are, respectively:

$$U_n = u_1 + u_2 + \cdots + u_n,$$
$$V_n = v_1 + v_2 + \cdots + v_n.$$

We shall now form the Cauchy product of the two series, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

The sequence $W_1, W_2, \cdots$, is formed where

$$W_n = w_1 + w_2 + \cdots + w_n$$
$$= u_1 V_n + u_2 V_{n-1} + \cdots + u_n V_1,$$

and

$$34) \qquad \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} U_i V_{n-i+1}.$$

Applying M and then E_r to (W_n), we have

$$35) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} W_k.$$

We shall now apply $\mathscr{M}$ to (U_n) and (V_n), and by 33) we obtain

$$z_n = 1/n \sum_{k=1}^{n} U_k V_{n-k+1}.$$

Applying E_r to (z_n), we have

$$36) \qquad z_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} U_h W_{k-h+1},$$

or

$$37) \qquad Z_n = E_r \mathscr{M}(U_n, V_n).$$

From 34), 35), 36), we see that

$$y_n = Z_n.$$

If now we show that $\lim_{n\to\infty} Z_n$ exists and equals $\sigma\tau$, our theorem is proved.

By Lemma 2, we have

$$38) \qquad E_r \mathscr{M}(U_n V_n) = \mathscr{M}[E_r(U_n), E_r(V_n)].$$

Let

$$\sigma_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} U_k,$$

$$\tau_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} V_k.$$

As $\sum u_n$, $\sum v_n$ are summable E_r,

$$\lim_{n\to\infty} \sigma_n = \sigma, \qquad \lim_{n\to\infty} \tau_n = \tau.$$

We have

$$39) \qquad \sigma_k = \sum_{p=1}^{k} \frac{(k-1)!}{(k-p)!(p-1)!} \frac{(r-1)^{k-p}}{r^{k-1}} U_p$$

similarly

$$40) \qquad \tau_{n-k+1} = \sum_{p=1}^{n-k+1} \frac{(n-k)!}{(n-k-p+1)!(p-1)!} \frac{(r-1)^{n-p-k+1}}{r^{n-p}} V_p.$$

From 37) and 38) we have

$$Z_n = \mathscr{M}[E_r(U_n), E_r(V_n)],$$

then substituting the values as given by 39) and 40) we find

$$Z_n = 1/n \sum_{k=1}^{n} \sigma_k \tau_{n-k+1}.$$

Obviously the conditions of Lemma 1 are satisfied and we state that

$$\lim_{n\to\infty} Z_n = \sigma\tau.$$

COROLLARY 1. *If $\sum u_n$ is summable E_r to the value σ and $\sum v_n$ is summable E_s to the value τ, then if $r > s$, the Cauchy product $\sum w_n$ is summable ME_r to the value $\sigma\tau$.*

This follows at once since the exponential mean becomes more inclusive as r increases.*

COROLLARY 2. *If $\sum u_n$, $\sum v_n$, and the Cauchy product $\sum w_n$ are summable † E_r to the values σ, τ, ω respectively, then $\omega = \sigma\tau$.*

If C_p denotes the Cesaro mean of order p then †

41) $$C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

where $\mathfrak{D}$ is the transformation

$$Z_n = \sum_{k=1}^{n} d_{n,k} x_k y_k,$$

with

$$d_{n,k} = \frac{(\mu+\nu+1)\,!\,(n-1)\,!\,(n+k-1)\,!\,(n-k+\nu)\,!}{\mu\,!\,\nu\,!\,(n-k)\,!\,(k-1)\,!\,(\mu+\nu+n)\,!}.$$

This transformation satisfies the conditions of Lemma 1; this consideration furnishes the usual proof ‡ of the theorem: *If $\sum u_n$ is summable C_μ to the value σ and $\sum v_n$ is summable to value τ, then the Cauchy product $\sum w_n$ is summable $C_{\mu+\nu+1}$ to value $\sigma\tau$.*

We proceed to generalize this result.

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* Hurwitz: *Report*, p. 27.

† Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift*, Vol. 15 (1922-23), p. 226.

Knopp gives a definition formed by applying E_2 successively p times. He proves for this definition the results stated in this corollary. But since $(E_2)^p = E_{2^p}$, his result is the special case in which r, instead of being any number greater than 1, is restricted to be an even integer.

‡ Chapman: "Non-Integral Orders of Summability of Series and Integrals," *Proceedings of London Mathematical Society*, Series 2, Vol. 9 (1910), p. 369.

THEOREM IX. *If $\sum u_n$ is summable $C_\mu E_r$ to the value σ and $\sum v_n$ is summable $C_\nu E_r$ to the value τ, then their Cauchy product is summable $C_{\mu+\nu+1} E_r$ to the value $\sigma\tau$.*

Since the transformation $\mathfrak{D}$ is permutable with M, we have

42) $$\mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)] = E_r \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

multiplying each side of 41) on the left by E_r, we obtain

43) $$E_r C_{\mu+\nu+1}(W_n) = E_r \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

From 42) and 43) we have

44) $$E_r C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)].$$

Thus we see from 44) that (W_n) is summable $E_r C_{\mu+\nu+1}$ to the value $\sigma\tau$.

COROLLARY 1. *If $\sum u_n$ is summable $C_\mu E_r$ to the value σ, and $\sum v_n$ is summable $C_\nu E_s$ to the value τ, then if $r > s$, the Cauchy product is summable $C_{\mu+\nu+1} E_r$ to the value $\sigma\tau$.*

CORNELL UNIVERSITY,
June, 1924.

The Eliminant of a Net of Curves.*

By Frank Morley.

A process was given by Sylvester † for writing in determinant form the condition that three homogeneous functions of three variables, of the same order, have a common zero, or that three curves of the same order, in a plane, have a common point.

The process is evidently not in final form, and Clebsch ‡ remarks "mais le caractère invariant n'en ressort pas clairement."

A memoir of A. L. Dixon § should also be mentioned, but as the method gives more than enough equations it is not applicable here. If instead of considering the discriminant as an invariant under the group of projections, we were considering it under the group of inversions, this memoir would be highly pertinent.

In what follows I obtain the eliminant as a determinant analogous to Sylvester's but obviously invariantive. Further, this determinant, properly bordered, is shown to be that form which gives the common point when it exists.

§ 1. *Salmon's connex.*

Salmon,‖ considering a plane curve and its tangent at a point x, wrote a connex which cuts out the remaining intersections of the tangent and the curve. It is of the form

$$A^3 x^{2(m-1)} y^{m-1}$$

when the curve is Ax^{m+1}. A^3 indicates that the coefficients are of the third degree in the coefficients A.

The connex naturally occurs in the theory of Abelian integrals of the third kind.¶

If now we remark that when the curve has a double point x, y becomes arbitrary, we have at once from Salmon's connex $m(m+1)/2$ curves of order $2m-2$ on the double point.

As all first polars are also on the double point, we have 3 curves of order m which give $3(m-1)m/2$ independent curves of order $2m-2$, so that in all we have a basis of $m(2m-1)$ curves of order $2m-2$, whence the dis-

* Read at the International Congress of Mathematicians, Toronto, 12 Aug., 1924.

† Salmon, *Higher Algebra*, § 91; Sylvester, *Works*, v. 1, p. 75 or *Phil. Mag.*, 1841.

‡ Clebsch-Lindemann. *Lectures*, v. 2. p. 13 of the French translation.

§ London Math. Soc., s. 2, v. 7, p. 49.

‖ Salmon, *Higher Plane Curves*, § 393 or *Phil. Mag.*, 1858; Cayley, *Works*, v. 4; Bramble, *Johns Hopkins Circulars*, July, 1915.

¶ Klein, "Abelsche Funktionen," *Math. Ann.*, v. 36 or *Works*, v. 3, p. 408.

criminant is written down as a determinant. What is necessary then is to adapt Salmon's connex to a net which is not a net of first polars.

§ 2. *The linear syzygy.*

Consider the sum of homogeneous products of 4 letters a, b, c, d,

$$H^n \equiv \Sigma a^n + \Sigma a^{n-1}b + \Sigma a^{n-2}b^n + \Sigma a^{n-2}bc + \cdots$$

and let this, when a is 0, become $H_a{}^n$.
Then evidently

$$H^n - aH^{n-1} = H_a{}^n.$$

Multiply the 4 such equations by numbers a_1, b_1, c_1, d_1, then

2.1) $$\Sigma a_1 . H^n - \Sigma a a_1 . H^{n-1} = \Sigma a_1 H_a{}^n.$$

Let now

$$a_1 = (ax) \mid \beta\gamma\delta \mid \qquad \text{and} \quad a = (ay)(\beta x)(\gamma x)(\delta x)$$
$$b_1 = -(\beta x) \mid \gamma\delta a \mid \qquad\qquad b = (ax)(\beta y)(\gamma x)(\delta x)$$
$$c_1 = (\gamma x) \mid \delta a\beta \mid \qquad\qquad c = (ax)(\beta x)(\gamma y)(\delta x)$$
$$d_1 = -(\delta x) \mid a\beta\gamma \mid \qquad\qquad d = (ax)(\beta x)(\gamma x)(\delta y),$$

where x and y are points of a plane.

Then $$\Sigma a_1 = 0 \text{ and } \Sigma a a_1 = 0.$$
Hence from 1) $$\Sigma a_1 H_a{}^n = 0,$$
that is $$\Sigma (ax)^{n+1} \mid \beta\gamma\delta \mid h_a{}^n = 0,$$

where $h_a{}^n$ is the sum of homogeneous products formed from $(\beta y)(\gamma x)(\delta x)$, $(\beta x)(\gamma y)(\delta x)$, $(\beta x)(\gamma x)(\delta y)$. If then we have 4 curves $(ax)^m$, $(\beta x)^m$, $(\gamma x)^m$, $(\delta x)^m$ there is the syzygy, linear in the coefficients,

$$\Sigma (ax)^m \mid \beta\gamma\delta \mid h_a{}^{m-1} = 0, \text{ or explicitly}$$
2.2) $$\Sigma^4 (\delta x)^m \mid a\beta\gamma \mid \Sigma (ax)^{\mu+\nu} (\beta x)^{\nu+\lambda} (\gamma x)^{\lambda+\mu} (ay)^{\lambda} (\beta y)^{\mu} (\gamma y)^{\nu} = 0$$

where $\lambda + \mu + \nu = m - 1$.

The same process gives a linear syzygy for 5 surfaces in a space, and so on.

§ 3. *The eliminant.*

Hence if the curves a, β, γ have a common point c, then since $(\delta c)^m$ is not 0,

3.1) $$\mid a\beta\gamma \mid h^{m-1} = 0,$$

where we have written h for h_δ.

This is the required connex. If we denote the curves a, β, γ by Ax^m, Bx^m, Cx^m the connex may be denoted by

$$ABCx^{2(m-1)}y^{m-1}.$$

When x is a common point, y is arbitrary. Thus we have $m(m+1)/2$ curves of order $2(m-1)$ on the common point, and the 3 given m-ics contribute $3(m-1)m/2$ curves of order $2(m-1)$, so that we have a basis of $m(2m-1)$ curves from which we eliminate.

The resulting determinant may be indicated by

3.2)
$$\Delta \equiv \begin{vmatrix} (m-1)m/2 & A \\ \text{“} & B \\ \text{“} & C \\ m(m+1)/2 & ABC \end{vmatrix}$$

§ 4. *A second form of the eliminant.*

The connex 3.1) is a combination of polars of y as to the Jacobian of the net, and as to the Jacobians of polars, as in Salmon's case (*Higher Plane Curves,* § 394). Other linear connexes which vanish at a common point are

4.1)
$$(\alpha\beta\gamma)(\alpha x)(\beta x)(\gamma x)h^{m-2}$$

or
$$ABCx^{2m-1}y^{m-2},$$

and generally

$$(\alpha\beta\gamma)(\alpha x)^r(\beta x)^r(\gamma x)^r h^{m-1-r}$$

or
$$ABCx^{2-m2+r}y^{m-1-r}.$$

For there is a syzygy

$$\Sigma(\alpha x)^m(\beta\gamma\delta)(\beta x)^r(\gamma x)^r(\delta x)^r h_\alpha^{m-1-r} = 0.$$

The series of connexes terminates with the Jacobian J, when $r = m-1$. When $r = m-2$ the connex is the polar of y as to J.

The form 4.1) also gives a basis. For it gives $(m-1)m/2$ curves of order $2m-1$, and the 3 given curves give $3m(m+1)/2$ curves of order $2m-1$, whence we have, for a common point, the eliminant as a determinant of $m(2m+1)$ rows,

$$\Delta' \equiv \begin{vmatrix} m(m+1)/2 & A \\ \text{“} & B \\ \text{“} & C \\ (m-1)m/2 & ABC \end{vmatrix}$$

§ 5. *The apolar forms.*

When we know that three curves have a common point, the next question is to find that point.

Let $(ax)^m$ be a point-form and $(a\xi)^n$ a line-form and let $n > \text{or} = m$. The polar of $(ax)^m$ as to $(a\xi)^m$ is by definition

$$(aa)^m(a\xi)^{n-m}.$$

When this vanishes, for an arbitrary line ξ, the forms are apolar (or incident). Apolarity means then $\binom{n-m+2}{2}$ conditions linear in either set of coefficients. Thus 3 conics have 3 independent apolar conics and a unique apolar cubic. Three cubics have 7 apolar cubics, 6 apolar quartics, 3 apolar quintics, and a unique apolar sextic. In the latter case if the cubics be $(ax)^3$, $(\beta x)^3$, $(\gamma x)^3$ and the sextic $(s\xi)^6$, there are 30 equations $(aa)^3(a\xi)^3 = 0$, but among them are three identities such as

$$(as)^3(\beta s)^3 = (\beta s)^3(as)^3.$$

To find this sextic, we may ask more generally for a sextic $(s\xi)^6$ such that the polars of $(ax)^3$, $(\beta x)^3$, $(\gamma x)^3$ as to it be given cubics, that is

$$\begin{gathered}(as)^3(s\xi)^3 \equiv \lambda(a\xi)^3 \\ (\beta s)^3(s\xi)^3 \equiv \mu(b\xi)^3 \\ (\gamma s)^3(s\xi)^3 \equiv \nu(c\xi)^3 \\ (s\xi)^6 = 0.\end{gathered}$$

We have then 31 equations from which to eliminate the 28 coefficients and λ, μ, ν. The resulting determinant has two factors,

$$(a\beta)^3(b\gamma)^3(ca)^3 - (a\gamma)^3(ba)^3(c\beta)^3$$

which vanishes if the problem is possible, and a sextic in ξ which is the apolar sextic.

Three quartics $(ax)^4 \cdots$ have a unique apolar ninic $(s\xi)^9$, for among the 3×21 equations are 9 identities

$$(as)^4(\beta s)^4(s\xi) \equiv (\beta s)^4(as)^4(s\xi)$$

And in general 3 forms $(ax)^m \cdots$ have a unique apolar form $(s\xi)^{3m-3}$, since

$$1 - 3\binom{m-1}{2} + 3\binom{2m-1}{2} - \binom{3m-1}{2} = 0.$$

Equally for ternary forms of different orders $(ax)^l$, $(\beta x)^m$, $(\gamma x)^n$ there is a unique apolar form $(s\xi)^{l+m+n-3}$ since

$$1 - \Sigma\binom{l-1}{2} + \Sigma\binom{m+n-1}{2} - \binom{l+m+n-1}{2} = 0.$$

And so in general in a flat F_d any $d+1$ forms have a unique apolar form of order $\Sigma m - d - 1$.

A point, on a curve $(ax)^m$, taken m or more times, is apolar to that curve. Thus when 3 curves have a common point c, $(c\xi)^{l+m+n-3}$ is the apolar form. But in this case $(c\xi)$ to any higher power is equally an apolar form. Thus taking the case of a net, there is in general no form of order $> 3m-3$ apolar to the net, unless a single condition is satisfied; this condition is in all cases the vanishing of the eliminant.

§ 6. *The bordered determinant.*

For clearness take a particular net, say a net of cubics. We have 3 cubics Ax^3, Bx^3, Cx^3 and the connex $ABCx^4y^2$ of § 3, which we will write $(\delta x)^4(\epsilon y)^2$. We have a unique apolar sextic, 3 independent apolar quintics, which are polars of points as to the sextic, and 6 independent apolar quartics which are polars of conics as to the sextic.

Consider these quartics $(s\xi)^4$. To name them we use the connex $(\delta x)^4(\epsilon y)^2$. The polar of $(s\xi)^4$ as to this quartic in x is

$$(\delta s)^4(\epsilon y)^2$$

a conic in y which we may call the conic associate with $(s\xi)^4$, and denote by $(\eta y)^2$. Thus an apolar quartic is defined by

$$(s\xi)^4 = 0, \quad (as)^3(s\xi) \equiv 0, \quad (\beta s)^3(\beta\xi) \equiv 0, \quad (\gamma s)^3(\gamma\xi) \equiv 0$$

$$\text{and} \quad (\delta s)^4(\epsilon y)^2 \equiv \rho(\eta y)^2.$$

Eliminating the 15 coefficients of the quartic and ρ from these 16 equations we have the bordered eliminant

$$\Delta(\xi^4, \eta^2) = \begin{array}{c|cc} & 15 & +\ 1 \\ \hline 3 & A & 0 \\ 3 & B & 0 \\ 3 & C & 0 \\ 6 & ABC & \eta^2 \\ 1 & \xi^4 & 0 \end{array}$$

This for given η is an apolar quartic. For given ξ it is the conic apolar to all associate conics of quartics on ξ.

If now $\Delta = 0$, then $\rho = 0$. The determinant factors into $(c\xi)^4$, where c is the common point, and a conic in η which is apolar to the associate conics of the apolar quartics, of which now that $(c\xi)^4$ is singled out there are only ∞^5.

§ 7. *The conic factor.*

When $\Delta = 0$ there is a common point c. It is to be shown that the conic factor of $\Delta(\xi^4 y^2)$ is then $(cy)^2$.

The 15 quartics have now an apolar quartic. There is thus an identity among them. That is $Ax^3 . A_1x + Bx^3 . B_1x + Cx^3 . Cx \equiv ABCx^4y^2$.

The fundamental syzygy (2.2) with $m = 3$ and δ a line on y, so that $(\delta y) = 0$, becomes

$$7.1) \qquad |\, a\beta\gamma \,|\, h^2(\delta x)^3 \equiv \Sigma(ax)^3 \,|\, \beta\gamma\delta \,|\, (\delta x)^2 \{(\beta y)^2(\gamma x)^2 + (\beta y)(\gamma y)(\beta x)(\gamma x) + (\beta x)^2(\gamma y)^2\}.$$

Let y be 1, 0, 0 and let this be a common point so that $a_0^3 = \beta_0^3 = \gamma_0^3 = \delta_0 = 0$. Then

$$7.2) \qquad |\, a\beta\gamma \,|\, h^2(\delta x) \equiv \Sigma(ax)^3 . |\, \beta\gamma\delta \,| \{\beta_0^2(\gamma x)^2 + \beta_0\gamma_0(\beta x)(\gamma x) + \gamma_0^2(\beta x)^2\}.$$

The cofactor of $(ax)^3$ contains the factor (δx). For writing $\delta_2 = x_1$, $\delta_1 = -x_2$ so that $|\, \beta\gamma\delta \,|$ is $\beta_0(\gamma x) - \gamma_0(\beta x)$, it is

$$\beta_0^3(\gamma x)^3 - \gamma_0^3(\beta x)^3$$

that is, 0. Hence (δx) divides each individual term in 7.2).

Thus the sought identity is

$$7.3) \qquad ABCx^4c^2 \equiv \Sigma Ax^3 . A_1x$$

and therefore when $\Delta = 0$ all apolar quartics $(s\xi)^4$ are apolar also to the form

$$ABCx^4c^2.$$

The associate conics are therefore all on the point c. Thus the conic factor is simply $(c\eta)^2$.

§ 8. *The unique apolar form.*

Thus when $\Delta = 0$, $\Delta(\xi^4, \eta^2) = (c\xi)^4(c\eta)^2$, and in particular $\Delta(\xi^4, \xi^2) = (c\xi)^6$. This is what the unique apolar sextic becomes when $\Delta = 0$, and since this unique sextic is of lower degree in the coefficients than Δ we infer that *the apolar sextic is in general*

$$\Delta(\xi^4, \xi^2).$$

Similarly if we border the second form of the eliminant we obtain $\Delta(\xi^5, \eta)$ which represents all apolar quintics. When $\Delta = 0$, the form becomes $(c\xi)^5(c\eta)$, and $\Delta(\xi^5, \xi)$ in another form of the apolar sextic, in par-

ticular when $\Delta = 0$, and therefore in general. So generally, by parallel argument, for a net of curves of order m, $\Delta(\xi^{2m-2}, \eta^{m-1})$ gives the apolar curves of order $2m-2$, $\Delta'(\xi^{2m-1}, \eta^m)$ gives the apolar curves of order $2m-1$, and when $\eta \equiv \xi$ either form becomes the unique apolar curve of order $3m-3$.

§ 9. *The case of several common points.*

The identical vanishing of the bordered eliminant means that the net has 2 common points. This case can be handled by bordering the eliminant twice. Thus for 3 conics we consider

$$\Delta(\xi^2\xi'^2, \eta\eta') \equiv \begin{array}{c|c|cc|} & 6 & 1 & 1 \\ \hline 1 & A & 0 & 0 \\ 1 & B & 0 & 0 \\ 1 & C & 0 & 0 \\ 3 & ABC & \eta & \eta' \\ 1 & \xi^2 & 0 & 0 \\ 1 & \xi'^2 & 0 & 0 \\ \hline \end{array}$$

When the net has two common points this equation in $\eta\eta'$, replacing $\overline{\eta\eta'}$ by x, in the join of the common points. And in general for n common points of any net we are to consider a determinant form of the eliminant, bordered n times.

On an Equation of Planar Motion.

By Frank Morley.

§ 1. *The equation.*

The points of a plane being named by complex numbers, let a point of the base circle, whose centre is 0 and radius 1, be denoted by t.

Then the general rotation, or displacement of a plane (say horizontal) over another, is given by

1.1) $$y = tx - a$$

where t and a are given.

We regard this not as an alibi but as an alias, y and x being the names in the two planes of two points in the same vertical.

From the ∞^3 equations (1) we now select ∞^1 by writing

$$a = f(t).$$

We have then for varying t an equation of motion,

1.2) $$y = tx - f(t).$$

§ 2. *Pins and slots.*

If in equation 2) we fix x, we have an equation which maps the base circle onto the points of the y-plane which successively coincide with x; that is, the path traced on the y-plane by a pin fixed in the x-plane.

Similarly if we fix a pin in the y-plane we have in (2) the map-equation of the path traced in the x-plane.

If we make slots in the planes along two such paths, we have two pins moving in two slots, and the motion is thus mechanically defined. The two pins may be in the one plane, and therefore the two slots in the other, or we may have a pin and a slot in each plane.

§ 3. *The cusp-loci, or centrodes.*

A curve expressed as a map of the base circle,

$$x = \phi(t),$$

has a cusp when there is a value of t for which

$$dx/dt = 0.$$

Thus the paths in the y-plane, for which x is given, have cusps when $dy/dt = 0$, or

3.1) $$x = f'(t).$$

and those in the x-plane, for which y is given, have cusps when $dx/dt = 0$, or

3.2) $$y = tf'(t) - f(t).$$

These are the centrodes, and the motion is the rolling of the one on the other. But this is, mechanically, no simplification. For instance the rolling of a circle on a circle is effected by the sliding motion of cycloidal teeth.

It is to be noted that when the curvatures of the centrodes at the common point are equal, all paths have a cusp. Thus cusps are not always on the centrodes.

§ 4. *Mechanical description of polynomic curves.*

We suppose the curve given as a map of the base-circle,

$$y = \phi(t).$$

If in $\phi(t)$ occurs a term ct^n, we replace the constant c by a variable x, so that

$$y = xt^n + \phi_1(t),$$

an equation of the form (1.2).

The slots in which move the pins $x = 0$, $y = 0$ are then

$$y = \phi_1(t), \qquad x = -\phi_1(t)/t^n.$$

These slots are in general simpler than the original curve, and to their description the same method may be applied.

We thus get rid of the terms in $\phi(t)$ which are of the form ct^n, and if $\phi(t)$ consists of a finite number of such terms the method will be practical.

The standard case is when y is a polynomial in t, say

$$y + a_1 t + a_2 t^2 + \cdots + a_n t^n = 0.$$

We replace this by the equation of motion

$$y_0 + a_1 t + a_2 t^2 + \cdots + x t^n = 0,$$

and the slots are

$$y + a_1 t + \cdots + a_{n-1} t^{n-1} = 0$$

and $$x + a_1/t^{n-1} + \cdots + a_{n-1}/t = 0$$

where $$y = xt^n.$$

§ 5. *Mechanical description of rational curves.*

If the curve be given by

$$y = \sum_1^n \frac{A_k}{a_k - t} \quad , \tag{5.1}$$

we write $(a_n - t)(\bar{a}_n - 1/\tau) = a_n\bar{a}_n - 1$.

Expressed by proper fractions in τ, y becomes

$$y = B_0 + B_1/\tau + \sum_1^{n-1} \frac{B_k}{b_k - \tau} \quad .$$

Moving the base-point to B_0, and replacing B_1 by x, we have the equation of motion in the form

$$y = x/\tau + \sum_1^{n-1} \frac{B_k}{b_k - \tau} \quad .$$

When $x = 0$, the slot is

$$y = \sum_1^{n-1} \frac{B_k}{b_k - \tau}$$

and when $y = 0$, the slot is

$$x = \sum_1^{n-1} \frac{B_k}{1 - b_k/\tau}$$

Repeating the process for each slot, we come to the case of two circular slots, or 3-bar motion (rational).

There remains the more general case when in the given expression of y in partial fractions there are repeated poles, as for instance a term $A/(a-t)^2$. Geometrically speaking, the curve has then a focus at infinity. The easy way is to allow inversions. After a suitable inversion, any rational curve is expressed in the form (5.1), the foci being now all finite.

Thus the general rational curve, after an inversion when necessary, can be mechanically described by two pins moving in two slots, the slots being simpler than the proposed curve.

An Extension of the Problem of the Elastic Bar.*

By H. T. Davis.

1. *The Problem of the Elastic Bar.* It is well known † that the problem of determining the transverse vibrations of an elastic bar leads to the study of an ordinary linear differential equation of fourth order with four linear boundary conditions, two of these conditions holding at one end of the bar and two at the other.

For example, if $w = f(x)$ is the initial elastic curve of a bar with fixed end points at $x = a$ and $x = b$, which starts vibrating without initial velocity, the ordinate of a point on the elastic curve satisfies the following partial differential equation:

$$\frac{\partial^4 W}{\partial x^4} + \phi(x) \frac{\partial^2 W}{\partial t^2} = 0,$$

and the following boundary conditions,

$$W(a, t) = W(b, t) = 0,$$
$$W'_x(a, t) = W'_x(b, t) = 0,$$

where $\phi(x)$ is a function of positive sign depending upon the physical properties of the bar.

If we assume that the solution is of the form

$$W(x, t) = u(x) \cos\sqrt{\lambda}\, t,$$

we are led to a study of the ordinary equation

$$\frac{d^4 u}{dx^4} - \lambda\phi(x) u = 0,$$

and the boundary conditions,

$$u(a) = u(b) = 0,$$
$$u'(a) = u'(b) = 0.$$

The problem of determining *characteristic functions* for this system and the expansion of $f(x)$ in terms of these functions is the problem of the elastic

* Presented to the American Mathematical Society, Dec. 29, 1922 and Dec. 26, 1924.

† See Lord Rayleigh's *Theory of Sound*, Vol. 1, chap. 8.

bar, which has been extensively studied. A. Davidoglou * in 1900 in a long memoir considered asymptotic developments in the problem. A. Myller † in his dissertation in 1906 computed Green's functions for various types of boundary conditions in the homogeneous case where $\phi(x) = 1$. More recently W. Stekloff and J. Tamarkine ‡ have applied methods of the former to the homogeneous case, and in 1913 C. F. Ward § recorded various numerical results for special cases of the problem. Other developments may be found in the application of the theory of integral equations to the subject, an account of which is given in *Die Integralgleichungen* by A. Kneser.‖

Now if we set

$$L(u) \equiv \frac{d^2}{dx^2}(p\,u'') + \frac{d}{dx}(q\,u') + ru,$$

it is clear that the problem presented by the elastic bar is generalized in the following differential system:

$$L(u) + \lambda k(x) u = 0,$$

$$U_i(u) \equiv \sum_{k=1}^{4} a_{ik}\, u^{(k-1)}(a) = 0, \quad (i = 1, 2) \tag{1}$$

$$U_j(u) \equiv \sum_{k=1}^{4} b_{jk}\, u^{(k-1)}(b) = 0, \quad (j = 3, 4)$$

where $p(x) \neq 0$, $k(x) \neq 0$ in the interval $a \leqq x \leqq b$.

In this paper we propose to develop conditions for the existence of characteristic numbers and characteristic functions for the system just written down under the additional assumption that the boundary conditions are self-adjoint.¶ A method due to G. D. Birkhoff ** will then be used to obtain the asymptotic form of the characteristic numbers. The novel element of the paper consists in associating with system (1) a square matrix of constants

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* *Annales de l'école normale supérieure*, 3rd series, Vol. 17 (1900), pp. 359-444. See in particular page 405.

† Dissertation, Göttingen (1906).

‡ *Rendiconti di Palermo*, Vol. 31 (1911), pp. 341-362. See also *Paris Comptes Rendus*, Vol. 154 (1912), pp. 269-271.

§ *Phil. Mag.* (6), Vol. 25 (1913), pp. 85-106.

‖ Second edition (Braunschweig), (1922), pp. 43-47.

¶ The idea of adjoint equations and adjoint systems of boundary conditions has been largely developed by Birkhoff and Bôcher. The reader is referred to Bôcher's *Leçons sur les méthodes de Sturm*, Paris, (1917), Chapter 2, for definitions used in this paper.

** *Transactions of the Amer. Math. Soc.*, Vol. 9 (1908), pp. 219-231.

which connects solutions of system (1) with solutions of the system adjoint to (1). Similar matrices exist for any problem where $L(u)$ is a self-adjoint expression of even order and much of the work that follows can be generalized for equations of higher order.

Boundary conditions in which $U_i(u)$ is a linear combination of $u, \cdots, u^{(n-1)}$ at one point only will be called *Sturmian boundary conditions,* since it was C. Sturm who in 1836 first studied such conditions associated with a homogeneous differential equation of second order.* It is important to notice that in the case of equations of order greater than three, different types of Sturmian boundary conditions appear. For example, in the case $n = 4$, we may have three conditions at one point and one at the other or, as in the case of the vibrating bar, we may have two conditions at each point. The problems presented by these two cases are very different from one another, for it is not difficult to show that *the boundary conditions in the first case can never be self-adjoint, while in the second case only two conditions on the coefficients are necessary for self-adjointness.*†

2. *A Preliminary Normalization.* We begin by considering the well-known Lagrange identity

$$\int_a^b [v\,L(u) - u\,L(v)]\,dx = P(u, v), \qquad (2)$$

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* *Journal de Mathématiques,* Vol. 1 (1836), pp. 106-186.

† This fact can be proved by means of a theorem due to D. Jackson. *Transactions of the Amer. Math. Soc.,* Vol. 17 (1916), pp. 418-424. Thus suppose that

$$\Pi = \begin{pmatrix} \pi_1 & \pi_3 \\ \pi_4 & \pi_2 \end{pmatrix}$$

is the matrix of the bilinear form $P(u, v)$ in equation (2), section 2. Let A_1 be the square matrix of the coefficients of n variables in $U_i(u)$, these coefficients being so chosen that A_1 is non-singular. Then A_2 will be the square matrix of the remaining n variables. If δ denotes the product $A_1^{-1} A_2$ and δ' the conjugate of δ then Jackson's theorem asserts:

"If the differential expression $L(u)$ is self-adjoint, the condition that the boundary conditions be self-adjoint is that the matrix

$$\delta'\Pi_1\delta - 2\Pi_4\delta + \Pi_2$$

be symmetric; if $L(u)$ is an anti-self-adjoint expression or differs from such an expression only in the term of order zero, the condition is that the matrix just written down be skew-symmetric."

The proof of the fact in question consists in showing that for any choice of a non-singular matrix A_1, we are led to the contradiction that the determinant of A_1 must be zero. The author hopes to be able to state later the corresponding facts for Sturmian systems of any order.

in which $P(u, v)$ is a non-singular bilinear form in the eight quantities $u(a), \cdots, u'''(a)\,; u(b), \cdots, u'''(b)\,;\ v(a), \cdots, v'''(a)\,; v(b), \cdots, v'''(b)$.

For the purpose of simplifying the coefficients of system (1) and reducing it to a normal form, we make in (2) the following transformation of both dependent and independent variables:

$$u(x) = \phi(x)y(x), \qquad x = \xi(t).$$
$$v(x) = \phi(x)w(x),$$

Indicating by a bar the value of a function of x after its transformation into a function of t, we shall have

$$\int_a^b [v\,L(u) - u\,L(v)]\,dx =$$

$$\int_{t_0}^{t_1} \bar{\phi}^2\bar{p}\left(\frac{d\xi}{dt}\right)^{-3}\bar{w}\left\{\frac{d^4\bar{y}}{dt^4} + \cdots + [\cdots + \lambda\bar{k}\left(\frac{d\xi}{dt}\right)^4]\bar{y}\right\}dt - \quad (3)$$

$$\int_{t_0}^{t_1} \bar{\phi}^2\bar{p}\left(\frac{d\xi}{dt}\right)^{-3}\bar{y}\left\{\frac{d^4\bar{w}}{dt^4} + \cdots + [\cdots + \lambda\bar{k}\left(\frac{d\xi}{dt}\right)^4]\bar{w}\right\}dt = P^*(\bar{y}, \bar{w}),$$

where P^* is the bilinear form in terms of the new variables $\bar{y}$ and $\bar{w}$.

We now notice that *the property of self-adjointness will be preserved providing the Lagrange identity remains invariant under the transformation.*

Therefore if we set,

$$\bar{\phi}^2\bar{p}\left(\frac{d\xi}{dt}\right)^{-3} = 1, \qquad \left(\frac{d\xi}{dt}\right)^4\bar{k} = \pm 1, \qquad (4)$$

where the ambiguous sign is determined from the sign of $\bar{k}$ so as to make the transformation real, it is clear that the resulting differential expression $L^*(y)$ will be self-adjoint and of the form

$$L^*(y) = \frac{d^4y}{dt^4} + \frac{d^2}{dt^2}\,(Q\,y') + R\,y + \lambda\,y. \qquad (5)$$

We must next see that the boundary conditions of system (1) have also remained self-adjoint under the transformation defined by (4).

Suppose that we employ the notation

$$U_i(u) = a_{i1}u_1 + \cdots + a_{i8}u_8 \quad (i = 1, \cdots, 4), \qquad (6)$$

$$V_j(u) = \beta_{j1}u_1 + \cdots + \beta_{j8}u_8 \quad (j = 5, \cdots, 8), \qquad (7)$$

where u_1 to u_8 are the eight values $u(a),\ u'(a), \cdots, u'''(a)\,; u(b), u'(b),$

$\cdots, u'''(b)$ and where U_i and V_j are connected by the identity *

$$P(u, v) \equiv \sum_{i,j=1}^{8} U_i V_j,$$

in which the first n V's and the last n U's are arbitrarily chosen.

From the definition of adjoint boundary conditions there must exist constants c_{jk} such that

$$V_j = c_{j1} U_1 + c_{j2} U_2 + \cdots + c_{j4} U_4, \quad (j = 5, \cdots, 8). \tag{8}$$

Upon substituting (6) and (7) in (8) and equating the coefficients of like terms we obtain four sets of eight equations of the form

$$c_{11}a_{1i} + c_{12}a_{2i} + \cdots + c_{14}a_{4i} = \beta_{ji}, \qquad (i = 1, \cdots, 8;\ j = 5, \cdots, 8). \tag{9}$$

If A denotes the matrix of this system of equations and B the augmented matrix

$$A = \begin{pmatrix} a_{11}, \cdots, a_{41} \\ \cdots\cdots \\ a_{18}, \cdots, a_{48} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11}, \cdots, a_{41}, -\beta_{j1} \\ \cdots\cdots\cdots \\ a_{18}, \cdots, a_{48}, -\beta_{j4} \end{pmatrix},$$

then a necessary and sufficient condition that the equation be consistent is that the ranks of A and B shall be equal.†

Suppose that the rank of A is r. By hypothesis the system is self-adjoint so that the c_{ij} in (9) actually exist. Consequently all of the determinants in B of order higher than r must vanish. If the condition be added that the U_i shall be linearly independent, then *the ranks of A and B must both be equal to n.*

From Leibnitz's formula it follows that (6) and (7) are submitted to the transformation

$$u^{(r)}(a) = \phi^{(r)}(a)y(a) + \binom{r}{1}\phi^{(r-1)}(a)y'(a) + \cdots + \binom{r}{r}\phi(a)y^{(r)}(a),$$
$$u^{(r)}(b) = \phi^{(r)}(b)y(b) + \binom{r}{1}\phi^{(r-1)}(b)y'(b) + \cdots + \binom{r}{r}\phi(b)y^{(r)}(b),$$

where $r = 0, \cdots, 3$.

Denote by C the matrix of this transformation. This matrix is clearly non-singular since its determinant reduces to the product of the principal diagonal which equals $[\phi(a)]^4[\phi(b)]^4$. By hypothesis on $p(x)$ and $k(x)$ this is different from zero.

Consequently B^*, the matrix of the transformed coefficients will be the

* See Bôcher's *Méthodes de Sturm, loc. cit.*, chapter 2.

† Bôcher, *Introduction to Higher Algebra*, New York (1922), p. 46.

following product,

$$B^* = C\,B.$$

Since B is of rank n by hypothesis and since the first matrix is non-singular, it follows that the rank of B^* must be n also.* A similar statement applies to $A\dagger = C\,A$.

The transformation of the dependent variable may be similarly treated since the determinant of the matrix of the linear transformation on A^* and B^* defined by it is easily shown to be $\left\{\left[\frac{d\xi(t_0)}{dt}\right]\left[\frac{d\xi(t_0)}{dt}\right]\right\}^{-6}$ which is non-singular from the hypothesis on $k(x)$.

These results may be summarized in the following theorem:

Theorem 1. *The self-adjointness property of both the differential equation and the boundary conditions of system* (1) *is unaltered by the transformations defined by equations* (4).

3. *The Fundamental Matrix and Its Properties.* Let us recall † that if $(u_1, \cdots, u_n)$ is a fundamental set of solutions of the general equation of nth order

$L(u) \equiv p_0(x)u^{(n)}(x) + p_1(x)u^{(n-1)}(x) + \cdots p_n(x)u(x) = 0$, and if W, the Wronskian, is the determinant

$$W = \begin{vmatrix} u_1 & u'_1 & u''_1 & \cdots & u_1^{(n-1)} \\ u_2 & u'_2 & u''_2 & \cdots & u_2^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_n & u'_n & u''_n & \cdots & u_n^{(n-1)} \end{vmatrix},$$

then $v_i = \frac{1}{p_0}\frac{\partial \log W}{\partial u_i^{(n-1)}}$, $i = 1, \cdots, n$, will be a set of linearly independent solutions of the adjoint equation. These solutions are called by Darboux the *adjoints* of the solutions u_i and satisfy the relations

$$\begin{aligned} &v_1u_1^{(j)} + v_2u_2^{(j)} + \cdots + v_nu_n^{(j)} = 0, \qquad j < n-1, \\ &v_1u^{(n-1)} + v_2u^{(n-1)} + \cdots + v_nu^{(n-1)} = 1/p_0. \end{aligned} \tag{10}$$

In case, however, that the equation is self-adjoint (or anti-self-adjoint),

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* Bôcher, *Higher Algebra*, p. 79, theorem 7.

† For an extensive development of this subject see G. Darboux, *La théorie des Surfaces*, Paris, (1889), part 2, book 4, chap. 5; also L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, Leipzig, (1895), Vol. 1, chap. 3.

it is clear that the v_i must be linear combinations of $(u_1, \cdots, u_n)$,

$$v_i = c_{i1}u_1 + c_{i2}u_2 + \cdots + c_{in}u_n, \quad (i = 1, \cdots, n). \tag{11}$$

The matrix of this transformation we shall designate by C, and we shall show that it has the following fundamental property:

Theorem 2. If $L(u) = 0$ is a self-adjoint (or anti-self-adjoint) equation of order n and if $(v_1, \cdots, v_n)$ are the adjoints of the solutions $(u_1, \cdots, u_n)$, then the matrix C of the substitution (11) *is either symmetric or skew-symmetric as n is odd or even.*

To prove this we first obtain from the n equations (11), n sets of n equations by differentiating v_i, $n-1$ times and from these we can solve for the values of the c_{ij} in terms of v_i and u_j and their $n-1$ derivatives. Thus, explicitly, we have

$$c_{ij} = [D_0 v_i + D_1 v'_i + \cdots + D_{n-1} v_i^{(n-1)}] \quad 1/W, \quad (i = 1, \cdots, n), \tag{12}$$

where D_k is the cofactor of $v_i^{(k)}$ in the determinant which we obtain by replacing the jth column of W by $v_i, v'_i, \cdots, v_i^{(n-1)}$.

But from equations (10) we observe that we have

$$D_{n-1} = p_0 v_j W.$$

By successive differentiation of this equation and the use of the differential equation in eliminating terms which involve $u_j^{(n)}$, we obtain the following expansions:

$$\begin{aligned}
D_0 &= (-1)^{n-1}[(p_0 v_j)^{(n-1)} - (p_1 v_j)^{(n-2)} + \cdots + (-1)^{n-1}(p_{n-1} v_j)]\, W, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_{n-3} &= [(p_0 v_j)'' - (p_1 v_j)' + (p_2 v_j)]\, W, \\
D_{n-2} &= -[(p_0 v_j)' - (p_1 v_j)]\, W, \\
D_{n-1} &= (p_0 v_j)\, W.
\end{aligned} \tag{13}$$

If we call Π the matrix of this substitution we observe that it is identical with the matrix of the bilinear form $P(u, v)$ in equation (2) which, in the self-adjoint case, is known to be symmetric when n is an odd number and skew-symmetric when n is an even number.*

When this substitution is made in (12), c_{ij} is expressed as a quadratic form in $v_i v_j$ whose matrix is symmetric when n is odd and skew symmetric when n is even. This proves the theorem.

We can obtain now by means of theorem 2 some elegant and important

* L. Schlesinger, *loc. cit.*, Vol. 1, pp. 70-71, 73-74.

identities satisfied by the second order minors of the Wronskian for the case $n = 4$, where we assume the differential equation in the normal form:

$$L(u) \equiv u^{(IV)} + p_2 u'' + p_2' u' + p_3 u = 0.$$

For convenience we shall designate by the notation

$$\begin{pmatrix} i & j \\ l & m \end{pmatrix} = \begin{pmatrix} u_l^{(i)} & u_l^{(j)} \\ u_m^{(i)} & u_m^{(j)} \end{pmatrix},$$

these minors of second order which are $\binom{4}{2}^2 = 36$ in number.

If we substitute the values of v_i $(i = 1, 2, 3, 4)$ given by (11) in identities (10), and those derived from (10) by differentiation, and at the same time recall theorem 2, we have at once the following remarkable relations between the minors: *

$$\begin{aligned}
&c_{12}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} = 0,\\
&c_{12}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} = 0,\\
&c_{12}\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 0 & 3 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} = 1,\\
&c_{12}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -1,\\
&c_{12}\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = 0,\\
&c_{12}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} + c_{24}\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} + c_{34}\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = -p_2.
\end{aligned} \tag{14}$$

That system (14), which may be looked upon as a non-homogeneous one in the six variables c_{ij}, is consistent for all choices of a fundamental system is readily seen as follows. The sixth order determinant of the c_{ij} consists of all of the minors of the second order of the Wronskian. If we call this determinant W_2, then by the theorem of Sylvester and Franke †

$$W_2 = W^{\binom{3}{1}} = W^3 \neq 0.$$

The explicit determination of the c_{ij}, however, is more easily accomplished as follows. Suppose that the fundamental system $(u_1, \cdots, u_4)$ has

* The linear dependence of the minors $\binom{0\ 1}{i\ j}$, which appears in the first line of (14) has long been known. In an article by G. H. Halphen, "Sur les invariants des équations différentielles linéaires du quatrième ordre," *Acta Mathematica*, Vol. 3, (1883), p. 329, they are shown to satisfy an equation of fifth order. See also Forsyth, *Philosophical Transactions*, Vol. 179 (1888), p. 455. It is believed that the other five relations and their connection with matrix C has not before been noted.

† G. Kowalewski, *Einführung in die Determinantentheorie*, Leipzig (1909), p. 104.

been uniquely determined by the following relations at some point $x = a$:

$$a_{i1}u(a) + a_{i2}u'(a) + a_{i3}u''(a) + a_{i4}u'''(a) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol which is equal to 1 when $i = j$ and equal to 0 when $i \neq j$, and where the determinant $D = |\, a_{ij} \,|$ is different from zero.

Then if A_{ij} is the cofactor of a_{ij} in the determinant D, it follows that

$$u_i^{(j-1)}(a) = A_{ij}/D \text{ and } W = (u_1, u'_2, u''_3, u'''_4) = D^3/D^4 = 1/D. \qquad (15)$$

Moreover, since

$$c_{11}u_1^{(j-1)} + c_{12}u_2^{(j-1)} + c_{13}u_3^{(j-1)} + c_{14}u_4^{(j-1)} = v_1^{(j-1)}, \quad (j = 1, 2, 3, 4),$$

we have

$$c_{1j} = (D_{j0}/W)v_1 + (D_{j1}/W)v_1' + (D_{j2}/W)v_1'' + (D_{j3}/W)v_1''',$$

where D_{jk} is the cofactor of $u_j^{(k)}$ in the determinant W.

But these cofactors are easily computed in terms of the a_{ij} if we make use of well known theorems on adjoint determinants.

Thus

$$D_{01}/W = D(D^2/D^3)a_{11} = a_{11}, \cdots, \; D_{jk}/W = a_{j,k+1},$$

from which it follows that

$$c_{ij} = a_{j1}v_i + a_{j2}v_i' + a_{j3}v_i'' + a_{j4}v_i'''.$$

Making use of equations (13) we can easily compute v_i. The values thus obtained are tabulated for convenience below:

i	1	2	3	4
$v(a)$	a_{14}	a_{24}	a_{34}	a_{44}
$v'(a)$	$-a_{13}$	$-a_{23}$	$-a_{33}$	$-a_{43}$
$v''(a)$	$a_{12}-p_2(a)a_{14}$	$a_{22}-p_2(a)a_{24}$	$a_{32}-p_2(a)a_{34}$	$a_{42}-p_2(a)a_{44}$
$v'''(a)$	$-a_{11}+p_2(a)a_{13}$	$-a_{21}+p_2(a)a_{23}$	$-a_{31}+p_2(a)a_{33}$	$-a_{41}+p_2(a)a_{43}$

4. *General Separation Theorems.* We now proceed to the examination of certain general results which follow from the relations developed in the last section.

Consider the following two normalized Sturmian systems:

$$L(u) + \lambda u \equiv u^{(IV)} + p_2u'' + p_2'u' + (p_3 + \lambda)u = 0,$$

$$\text{(I)} \quad A_1(u) = A_2(u) = 0, \quad B_1(u) = B_2(u) = 0, \qquad \text{(II)} \quad A_3(u) = A_4(u) = 0, \quad B_1(u) = B_2(u) = 0, \qquad (16)$$

where the $A_i(u)$ are linear conditions at the point $x = a$ and the $B_i(u)$ are linear conditions at the point $x = b$.

If we assume that the determinant $| a_{ij} |$ is different from zero, then we can choose the fundamental set of solutions so that

$$A_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol, and the characteristic numbers of the two systems will appear as roots of

$$\Delta_1 = \begin{vmatrix} B_{13} & B_{14} \\ B_{23} & B_{24} \end{vmatrix}, \text{ and } \quad \Delta_2 = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix},$$

respectively.

Suppose, further, that in matrix C we have $c_{13} = c_{14} = c_{23} = c_{24} = 0$ and $c_{12} \neq 0$, $c_{34} \neq 0$. These are conditions upon the coefficients of $A_i(u)$, but they are consistent and can be satisfied in an infinite number of ways.

It is easily seen that Δ_2 can be expanded as follows: *

$$\Delta_2 = \gamma_{34}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \gamma_{24}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + \gamma_{23}\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} + \gamma_{14}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \gamma_{13}\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + \gamma_{12}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$

where γ_{ij} is the second order determinant formed by striking out the ith and jth columns in the following matrix:

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}.$$

Now if each column of the identities (14) be multiplied by the proper γ_{ij} and the resulting equations added together, we see that Δ_1, will equal Δ_2 multiplied by a non-vanishing constant providing the following condition is satisfied:

$$\gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0. \tag{17}$$

This result is stated in the following theorem, which is interesting in that it has no analogue in Sturmian systems of second order.

THEOREM 3. *The characteristic numbers of systems (I) and (II), (16), are identical providing the following conditions are satisfied:*

* Kowalewski, *Determinantentheorie, (loc. cit.)*, p. 72.

$$1)\quad c_{13} = c_{14} = c_{23} = c_{24} = 0,$$
$$2)\quad |\, a_{ij} \,| \neq 0,$$
$$3)\quad \gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0.$$

As an example consider the following systems:

$$\text{(I)}\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u'(0) = u''(0) = 0,\\ &u(\pi) = u'(\pi) = 0, \end{aligned} \qquad \text{(II)}\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u(0) = u'''(0) = 0,\\ &u(\pi) = u'(\pi) = 0. \end{aligned}$$

The conditions of the theorem are seen to be satisfied. We choose for a fundamental set of solutions the following:

$$u_1 = e^{\rho x},\quad u_2 = e^{-\rho x},\quad u_3 = \sin \rho x,\quad u_4 = \cos \rho x, \tag{18}$$

where $\rho = \sqrt[4]{\lambda}$. In terms of these functions we can now easily compute the two determinants Δ_1 and Δ_2, whose zeros are the characteristic numbers for the two systems (I) and (II). These determinants are found to be identical since

$$\Delta_1 = \Delta_2 = 4\lambda \sinh \rho\pi \sin \rho\pi,$$

and the characteristic numbers are $\lambda = \pm 1^4, \pm 2^4, \pm 3^4, \cdots$.

We next derive a separation theorem for the characteristic numbers of the following Sturmian systems:

$$u^{(IV)} + p_2 u'' + p_1' u' + (p_3 + \lambda)u = 0,$$
$$\text{(I)}\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &B_1(u) = B_2(u) = 0, \end{aligned} \qquad \text{(II)}\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &B_3(u) = B_4(u) = 0. \end{aligned} \tag{19}$$

In order to discover the conditions on the coefficients of the two systems so that the characteristic numbers of one will alternate or coincide with the characteristic numbers of the second, we shall make use of the following theorem recently published by the author: *

THEOREM. *If the existence of a sequence of characteristic numbers for problem* (I) *has been established, then a sequence of characteristic numbers for problem* (II), *alternating or coincident with the characteristic numbers of problem* (I), *exist provided*

$$\int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]\, dx \neq 0, \qquad (\lambda_0 \leqq \lambda \leqq \lambda_m) \tag{20}$$

* *Bulletin of the Amer. Math. Soc.*, Vol. 28, (1922), pp. 390-394.

where Γ_1 *and* Γ_2 *are the Green's functions belonging to problems* (I) *and* (II) *respectively.*

Let us now recall that we can write the Green's function of a differential equation and its boundary conditions in the form *

$$\Gamma(x, t) = C_1 u_1(x) + \cdots + C_n u_n(x) + g(x, t),$$

where $g(x, t)$ is a function defined as follows:

$$g(x, t) = \pm \frac{1}{2W} \begin{vmatrix} u_1(x), & u_1^{(n-2)}(t), & u_1^{(n-3)}(t), & \cdots, & u_1(t) \\ u_2(x), & u_2^{(n-2)}(t), & u_2^{(n-3)}(t), & \cdots, & u_2(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n(x), & u_n^{(n-2)}(t), & u_n^{(n-3)}(t), & \cdots, & u_n(t) \end{vmatrix}.$$

Here the plus sign is to be used when $x < t$ and the negative sign when $x > t$. W is the Wronskian or determinant of the fundamental set of solutions and their first $n-1$ derivatives.

We now notice that, by means of equations (10), it is possible to write $g(x, t)$ in the form

$$g(x, t) = \pm \tfrac{1}{2}[u_1(x)v_1(t) + \cdots + u_n(x)v_n(t)],$$

where the ambiguous sign is to be determined as before.

In order to determine the constants in the Green's function, we substitute $\Gamma(x, t)$ in the boundary conditions, from which it is immediately apparent that the C_i are expressible as linear forms in the $v_i(t)$. Hence the difference of the two Green's functions in the theorem stated above will be a bilinear form in the variables $u_i(x)$, $v_i(x)$, i. e.

$$\Gamma_1(x, x) - \Gamma_2(x, x) = \sum_{ij}^{n} f_{ij} u_i(x) v_j(x). \tag{21}$$

Now if we denote by F the matrix of this bilinear form and by C the matrix of the linear substitution (11), we can change (21) by means of (11) into a quadratic form in $u_i(x)$, $u_j(x)$ whose matrix will be

$$T = F\,C.$$

By a fundamental theorem on Green's functions, since $\Gamma_1(x, t)$ and $\Gamma_2(x, t)$ both belong to self-adjoint systems *they are both symmetric functions*

* J. Horn, *Theorie der partiellen Differentialgleichungen*, Leipzig, (1910), p. 243.

*of x and t and consequently their difference is a symmetric function.**

It follows at once, then, since T is the matrix of a bilinear form in $u_i(x)$, $u_j(t)$ which is symmetric that T is itself a symmetric matrix. Consequently when we consider the question of whether or not the quadratic form (21) is definite, we need only to examine the signs of the principal minors of the determinant of T and apply to them a well known criterion.†

We next proceed to the calculation of the matrix T for the special Sturmian systems (19). Suppose, for simplicity, that matrix C, by the choice of a proper fundamental system, is the following:

$$C = \begin{bmatrix} 0 & 0 & 0 & c_{14} \\ 0 & 0 & c_{23} & 0 \\ 0 & -c_{23} & 0 & 0 \\ -c_{14} & 0 & 0 & 0 \end{bmatrix}.$$

We see explicitly from the values of c_{ij} computed in the preceding section that this imposes upon the coefficients of $A_1(u)$, $A_2(u)$ the single condition $c_{12} = 0$.

If for brevity we use the notation

$$T_{ij} = \begin{vmatrix} B_{1i} & B_{1j} \\ B_{2i} & B_{2j} \end{vmatrix} \quad \text{and} \quad S_{ij} = \begin{vmatrix} B_{3i} & B_{3j} \\ B_{4i} & B_{4j} \end{vmatrix},$$

then the matrices of the Green's functions for the two systems (19) will be M_1 and M_2 where

$$M_1 = \begin{bmatrix} 0, & 0, & 0, & c_{14} \\ 0, & 0, & c_{23}, & 0 \\ 0, & c_{23}, & -2c_{23}T_{24}/T_{34}, & -2c_{14}T_{14}/T_{34} \\ c_{14}, & 0, & 2c_{23}T_{34}/T_{34}, & 2c_{14}T_{13}/T_{34} \end{bmatrix} \tag{22}$$

and where M_2 is an identical array except that T_{ij} has been replaced throughout by S_{ij}.

From (22) we are able to write down the conditions for the self-adjointness of the general Sturmian system of fourth order. Since matrix M_1 must be symmetric we shall have

$$c_{14}T_{14} + c_{23}T_{23} = 0,$$

the condition for which is (17). *Hence the general Sturmian system of fourth order in normal form will be self-adjoint providing its coefficients satisfy the conditions:*

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* Bôcher, *Méthodes de Sturm, loc. cit.*, pp. 104-105.

† Kowalewski, *Determinantentheorie, loc. cit.*, pp. 220 and 238.

$$c_{12} = 0, \qquad (23)$$
$$\gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0.$$

Returning to (22) we observe that

$$T = M_1 - M_2,$$

so that we have explicitly,

$$T = \begin{bmatrix} 0, & 0, & 0, & 0, \\ 0, & 0, & 0, & 0, \\ 0, & 0, & -2c_{23}T_{24}/T_{34} + 2c_{23}S_{24}/S_{34}, & -2c_{14}T_{14}/T_{34} + 2c_{14}S_{14}/S_{34} \\ 0, & 0, & 2c_{23}T_{23}/T_{34} - 2c_{23}S_{23}/S_{34}, & 2c_{14}T_{13}/T_{34} - 2c_{14}S_{13}/S_{34} \end{bmatrix}$$

In order to prove the definiteness of the form whose matrix is T, we must show that the following determinant is either positive or zero:

$$\Delta = \begin{vmatrix} -S_{34}T_{24} + T_{34}S_{24}, & -T_{14}S_{34} + S_{14}T_{34} \\ S_{34}T_{23} - T_{34}S_{23}, & T_{13}S_{34} - T_{34}S_{13} \end{vmatrix},$$

$$= -[S^2_{34}(T_{24}T_{13} - T_{14}T_{23}) + S_{34}T_{34}(-S_{24}T_{13} - T_{24}S_{13} + T_{14}S_{23} + T_{23}S_{14}) + T^2_{34}(S_{13}S_{24} - S_{14}S_{23})].$$

If we now expand the determinant $|B_{ij}|$ by Laplace's development according to the first two rows we shall have

$$|B_{ij}| = |b_{ij}|\,|u_{ij}| = |b_{ij}|\,W, \qquad (24)$$
$$= T_{12}S_{34} - T_{13}S_{24} + T_{14}S_{23} + T_{23}S_{14} - T_{24}S_{13} + T_{34}S_{12}.$$

Also, replacing in (24) the last two rows by the first two and expanding as before by Laplace's development, we may prove that

$$T_{12}T_{34} - T_{13}T_{24} + T_{14}T_{23} = 0,$$

and similarly that

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0.$$

When these values are substituted above we shall have

$$\Delta = -[S^2_{34}T_{12}T_{34} + S_{34}T_{34}(|b_{ij}|\,W - T_{12}S_{34} - T_{34}S_{12}) + T^2_{34}S_{12}S_{34}],$$

$$= -|b_{ij}|\,W\,T_{34}S_{34}.$$

Hence $\Delta = 0$, and the form is definite, provided that

$$|b_{ij}| = 0.$$

These results may be summarized in the following theorem:

THEOREM 4. *The characteristic numbers of systems (I) and (II), (19), either coincide or alternate with one another provided that their coefficients satisfy the following conditions:*

$$1)\ c_{12} = 0,$$
$$2)\ \gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0.$$
$$3)\ |\, b_{ij}\,| = 0.$$

A simple example will illustrate the theorem. Consider the systems:

$$\text{(I)}\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u(0) = u'(0) = 0,\\ &u(\pi) = u'(\pi) = 0; \end{aligned} \qquad \text{(II)}\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u(0) = u'(0) = 0,\\ &u(\pi) = u''(\pi) = 0. \end{aligned}$$

The coefficients are seen to satisfy the conditions of the theorem.

If we choose as a fundamental system, the one given by (18), we can compute the two determinants Δ_1 and Δ_2, whose zeros are the characteristic numbers of the two systems (I) and (II). These are found to be

$$\Delta_1 = 1 - \cos h \sqrt[4]{\lambda\pi} \cos \sqrt[4]{\lambda\pi},$$
$$\Delta_2 = \cos h \sqrt[4]{\lambda\pi} \sin \sqrt[4]{\lambda\pi} - \sin h \sqrt[4]{\lambda\pi} \cos \sqrt[4]{\lambda\pi}.$$

From these expressions we see that the characteristic numbers for the two systems will be

$$\lambda_n = [\tfrac{1}{2}(2n-1) + \epsilon_n]^4, \quad \lambda'_n = [\tfrac{1}{4}(4n-3) + \epsilon'_n]^4,$$

where ϵ_n and ϵ'_n are numbers which become vanishingly small for large values of n. The separation of the characteristic numbers by one another is thus explicitly seen for large values of n and may be verified by a graph for small values of n.

5. *Asymptotic Distribution of the Characteristic Numbers.* We can make use of theorem 4 to obtain an existence theorem for the general Sturmian system of fourth order. For this purpose consider the following systems:

$$u^{(IV)} + p_2 u'' + p'_2 u' + (p_3 + \lambda)u = 0,$$

$$\text{(I)}\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &B_1(u) = B_2(u) = 0, \end{aligned} \qquad \text{(II)}\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &u(b) = u'(b) = 0. \end{aligned} \qquad (25)$$

Choose for a fundamental set of solutions the one for which at the point $x = a$, the following conditions are satisfied,

$$u_i^{(j-1)}(a) = \delta_{ij}, \tag{26}$$

where δ_{ij} is the Kronecker symbol.

Then in system (II), (25), the A_{ij} are uniquely determined and the characteristic numbers will be zeros of the following function:

$$F(b,\lambda) = E_{12}\begin{pmatrix}0&1\\1&2\end{pmatrix} - E_{13}\begin{pmatrix}0&1\\1&3\end{pmatrix} + E_{14}\begin{pmatrix}0&1\\1&4\end{pmatrix} + E_{23}\begin{pmatrix}0&1\\2&3\end{pmatrix} - E_{24}\begin{pmatrix}0&1\\2&4\end{pmatrix} + E_{34}\begin{pmatrix}0&1\\3&4\end{pmatrix},$$

where E_{ij} is the second order determinant obtained by striking out the ith and jth columns in the matrix formed from the coefficients of $A_1(u)$, $A_2(u)$.

Furthermore the value of $F(x,\lambda)$, from conditions (26) is known for the point $x = a$, and by successive differentiation the values of $\left.\frac{dF}{dx}\right|_{x=a}, \cdots, \left.\frac{d^4F}{dx^4}\right|_{x=a}$ can be computed also. The explicit forms of these values are important in application and are recorded below:

$$F(a) = E_{12},\ F'(a) = -E_{13},\ F''(a) = E_{14} + E_{23},\ F'''(a) = E_{13}p_2(a) - E_{12}p'_2(a) - 2E_{24},\ F^{(IV)}(a) = 2E_{12}\lambda - 3p_2(a)E_{23} + [2p_3(a) - p_2''(a)]E_{12} + 2p_2(a)E_{13} - p_2(a)E_{14} + 2E_{34}. \tag{27}$$

But it is known that $F(x,\lambda)$ satisfies the following differential equation of fifth order: *

$$\frac{d^5F}{dx^5} + 2p_2\frac{d^3F}{dx^3} + 3p'_2\frac{d^2F}{dx^2} + (3p_2'' + p_2^2 - 4p_3 - 4\lambda)\frac{dF}{dx} + (p_2''' + p_2p'_2 - 2p'_3)F = 0. \tag{28}$$

Consequently, if we apply the results of theorem 4 to systems (I) and (II), (25), we can reduce the existence problem for the general self-adjoint Sturmian system of fourth order to the solution of a Cauchy problem of fifth order. Thus we have:

THEOREM 5. *Given an integral $F(x,\lambda)$ of equation* (28) *which is uniquely defined by the values* (27). *If the coefficients of system* (I), (25) *satisfy the following conditions:*

1) $c_{12} = 0$, 2) $\gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0$,
3) $(b_{13}b_{24} - b_{14}b_{23}) = 0$,

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* *Halphen, loc. cit.*, p. 329.

and if $F(b, \lambda)$ vanishes m times in the λ-interval, $\Lambda_1 < \lambda < \Lambda_2$, then there exist at least $m-1$ characteristic numbers of system (I) *which either alternate or coincide with the m λ-roots of $F(b, \lambda)$ in the interval (Λ_1, Λ_2).*

From this theorem the asymptotic distribution of the characteristic numbers for the general Sturmian system is readily deduced. For this purpose we make use of an important theorem due to Birkhoff,* which asserts that for values of ρ in a region S, the asymptotic form of the solutions of the equation

$$\frac{d^n u}{dx^n} + \rho a_{n-1}(x, \rho)\, \frac{d^{n-1} u}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho) u = 0,$$

where $a_i(x, \rho) = \sum_{j=0}^{\infty} a_{ij}(x) \rho^{-j}$, are given by

$$u_i^{(j)} = u_i^{(j)}(x, \rho) + e^{\rho \int_a^x \omega_i(t)\, dt}\, \frac{E_i}{\rho^{m-j}}, \quad \begin{matrix} (i = 1, 2, \cdots, n, \\ (j = 0, 1, 2, \cdots, n-1, \\ (m \text{ arbitrary}, \end{matrix}$$

in which $u_i(x, \rho) = e^{\rho \int_a^x \omega_i(t)\, dt} \sum_{j=0}^{m-1} \frac{u_{ij}(x)}{\rho^j}$, $u_{i0}(x) \neq 0$ for $a \leqq x \leqq b$.

The ω_i are roots of the equation

$$\omega^n + a_{n-1\,0}(x)\omega^{n-1} + \cdots + a_{00}(x) = 0,$$

and the E_j are functions of ρ and x bounded for large values of ρ.

We now apply this theorem to equation (28). Two cases are to be considered according to whether we set $\lambda = -\rho^4$ or $\lambda = +\rho^4$.

First Case. If we choose $\lambda = -\rho^4$, then $a_1 = 4 + \frac{f(x)}{\rho^4}$, so that the values ω_i satisfy the equation

$$\omega^5 + 4\omega = 0. \tag{29}$$

A fundamental set of solutions of equation (28) will then be

$$F_i(x) = e^{\omega_i(x-a)\rho}\left[1 + \frac{F_{i1}(x)}{\rho} + \frac{F_{i2}(x)}{\rho^2} + \cdots\right], \quad (i = 1, 2, \cdots, 5) \tag{30}$$

where $\omega_1 = 0$, $\omega_2 = 1 + i$, $\omega_3 = 1 - i$, $\omega_4 = -1 + i$, $\omega_5 = -1 - i$.

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* G. D. Birkhoff, *Transactions of the Amer. Math. Soc.*, Vol. 9, (1908), p. 225.

In terms of these functions $F(b)$ may be written:

$$F(b) = \frac{1}{W(b)} \sum_{i=1}^{5} \begin{vmatrix} F_1(a), \cdots, F_{i-1}(a), F(a), F_{i+1}(a), \cdots, F_5(a) \\ F'_1(a), \cdots, F'_{i-1}(a), F'(a), F'_{i+1}(a), \cdots, F'_5(a) \\ \cdots\cdots\cdots\cdots\cdots \\ F_1^{(IV)}(a), \cdots, F_{i-1}^{(IV)}(a), F^{(IV)}(a), \\ F_{i+1}^{(IV)}(a), \cdots, F_5^{(IV)}(a) \end{vmatrix} F_i(b),$$

where $W(b)$ is the Wronskian of the fundamental system.

From (27) we see that all of the $F^{(j-1)}(a)$ are independent of λ except $F^{(IV)}(a)$ which is linear in λ and equal to $2F(a)\lambda +$ a constant. We now observe that in (31) for large positive values of ρ the dominant terms will be those containing $F_2(b)$ and $F_3(b)$, and similarly for large negative values of ρ the dominant terms will be those containing $F_4(b)$ and $F_5(b)$, provided the coefficients of these terms do not vanish. That these coefficients are, indeed, always different from zero can be seen explicitly by expanding them in terms of $1/\rho$ and retaining only those coefficients of the $F^{(j-1)}(a)$ which are of lowest degree in $1/\rho$. The coefficients of $F_i(b)$, $i > 1$, will then be asymptotic to

$$\pm \rho^{10} \{2F(a)D_{4i} + \frac{F'(a)D_{1i}}{\rho} + \frac{F''(a)D_{2i}}{\rho^2} + \cdots\},$$

where the D_{ki} are the cofactors of the element in the kth row and $(i-1)$th column of the following determinant:

$$D = \begin{vmatrix} \omega_2 & \omega_3 & \omega_4 & \omega_5 \\ \omega_2^2 & \omega_3^2 & \omega_4^2 & \omega_5^2 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_2^4 & \omega_3^4 & \omega_4^4 & \omega_5^4 \end{vmatrix}.$$

Since none of these cofactors are zero and since, moreover, not all of the values $F^{(j-1)}(a)$ can be zero, because $F(b)$ would then be identically zero, it follows that none of the coefficients of $F_i(b)$, $i > 1$, can vanish.

We next examine several special cases. Suppose that $F(a) \neq 0$. It will be found by a simple computation and by combining the second and third terms of (31), that the dominant terms in $F(b)$ for positive values of ρ, will be

$$A\, e^{(b-a)\rho} \cos \rho(b-a), \quad A \neq 0.$$

From the ρ-zeros of this function we can compute the asymptotic form of the characteristic numbers, thus obtaining

$$\lambda_i \sim -\left\{\frac{2n-1}{2(b-a)}\,\pi\right\}^4,$$

for n sufficiently large.

In order to obtain the λ_i which correspond to negative values of ρ, we combine the fourth and fifth terms of (31). These will be found asymptotic to $-\dfrac{2n-1}{2(b-a)}\,\pi$ so that no new values of λ_i are introduced.

In a similar way we examine the case where $F(a)=0$, $F'(a)\neq 0$. The dominant function is then of the form

$$A\,e^{(b-a)\rho}\sin\{\rho(b-a)+\pi/4\},\quad A\neq 0,$$

from which it follows that

$$\lambda_i \sim -\left\{\frac{4n-1}{4(b-a)}\,\pi\right\}^4.$$

Similarly, if $F''(a)$ is the first value different from zero, we obtain $\lambda_i \sim -\{n\pi/(b-a)\}^4$. The case where $F'''(a)$ is the first non-vanishing value gives the same asymptotic form of the characteristic numbers as for $F(a)=0$, $F'(a)\neq 0$, and the case where $F^{(IV)}(a)\neq 0$, corresponds to the first case.

Second Case. If we choose $\lambda=\rho^4$, then the values of ω_i are the roots of $\omega^5-4\omega=0$, or $\omega_1=0$, $\omega_2=\sqrt{2}$, $\omega_3=-\sqrt{2}$, $\omega_4=\sqrt{2i}$, $\omega_5=-\sqrt{2i}$.

The same argument applies as in the preceding case except that the dominant terms are now either $e^{\sqrt{2}\rho(b-a)}$ for positive values of ρ or $e^{-\sqrt{2}\rho(b-a)}$ for negative values of ρ. Hence there will not exist characteristic numbers for large values of ρ and the λ_i-sequence is finite.

These results may be summarized as follows:

THEOREM 6. *If the coefficients of system (1), (25), satisfy the conditions of theorem 5, then there will exist for negative values of λ an infinite sequence of characteristic numbers, separated by, or coincident with, the following values when n is taken sufficiently large:*

(a) $\lambda_i \sim -\left\{\dfrac{2n-1}{2(b-a)}\,\pi\right\}^4$, *if* $F(a)\neq 0$; *or if* $F^{(IV)}(a)\neq 0$ *and all the preceding values zero.*

(b) $\lambda_i \sim -\left\{\dfrac{4n-1}{4(b-a)}\,\pi\right\}^4$, *if* $F(a)=0$, $F'(a)\neq 0$; *or if* $F'''(a)\neq 0$ *and all preceding values zero.*

(c) $\lambda_i \sim -\left\{\dfrac{n}{b-a}\,\pi\right\}^4$, *if* $F''(a)\neq 0$ *and all preceding values zero.*

For positive values of λ, the number of characteristic values is always finite.

As an illustration consider the characteristic numbers for the system:

$$u^{(IV)} - \lambda u = 0,$$
$$u(0) = u'(0) = 0, \quad u(\pi) = u''(\pi) = 0.$$

By a simple computation we have

$$F(0) = F'(0) = F''(0) = F'''(0) = 0, \quad F(0) = 2.$$

Hence the characteristic numbers must alternate or coincide with the values $+\left\{\frac{2n-1}{2}\right\}^4$, a result consistent with that obtained in the last example of the preceding section.

INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA.

On the Sylow Subgroups of the Symmetric and Alternating Groups.

By Louis Weisner.

1. *Introduction.* It is the object of this paper to exhibit a formula for calculating the number of Sylow subgroups of each order in the symmetric and alternating groups of degree n. Let p be a prime factor of $n!$ and let p^m be the highest power of p that divides $n!$; further let n be written in the scale of p in the form

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.$$

The number of Sylow subgroups of order p^m in the symmetric group of degree n will be shown to be $\dfrac{n!}{a_0!\ a_1! \cdots a_k!\ p^m (p-1)^m}$.
Incidentally other properties of the Sylow subgroups of the symmetric group will be noted.

The number of Sylow subgroups of order p^m in the alternating group of degree n is evidently the same as the number in the symmetric group if $p > 2$. It will be shown that for $n > 5$, the alternating group contains as many Sylow subgroups of even order as the symmetric group.

2. *An Invariant Abelian Subgroup.* The substitutions

$$s_1 = (1, 2, \cdots, p), \quad \overline{s_2 = (p+1, p+2, \cdots, 2p)}, \cdots$$
$$\overline{s_{n_1} = (n_1 - 1\ p + 1, n_1 - 1\ p + 2, \cdots, n_1 p)},$$

where, with the notation of § 1,

$$n_i = a_i p + a_{i+1} p^2 + \cdots + a_k p^{k-i} \qquad (i = 1, 2, \cdots, k),$$

clearly generate an abelian subgroup I of order p^{n_1} and type $(1, 1, \cdots, 1)$. We proceed to prove that *I is invariant under every Sylow subgroup H of G that contains I,* where G denotes the symmetric group of degree n.

A conjugate I' of I under G contains conjugates of $s_1, s_2, \cdots, s_{n_1}$. At least one of these conjugates must affect one of the symbols $1, 2, \cdots, p$. Let $t = (1, i_2, \cdots, i_p)$ be such a substitution of I', and for definiteness suppose t affects the symbol 1. If H contains I and I', (s_1, t) is in H and hence is of order a power of p, say p^a. Now (s_1) is transitive in the symbols

$1, 2, \cdots, p$, and (t) is transitive in the symbols $1, i_2, \cdots, i_p$. It follows that (s_1, t) is transitive in the symbols which it affects. Hence p^a is divisible by the degree of (s_1, t) which, being less than $2p$, must equal p. Hence s_1 and t affect the same p symbols. Since the symmetric group on p letters involves no subgroup of order p^2, (s_1, t) is of order p. Hence s_1 is a power of t and is contained in I'. Similarly I' contains $s_2, \ldots, s_{n_1}$. Hence $I' = I$.

3. *Number of Conjugates of I.* We proceed to determine the number of conjugates of I under G by calculating the number of ways of selecting the cyclic subgroups, each involving a single cycle of p letters, of which a conjugate of I is the direct product. Denoting by ${}_xP_y$ the number of permutations of x things taken y at a time, it is clear that s_1 may be chosen in ${}_nP_p/p$ ways, and hence (s_1) may be chosen in ${}_nP_p/p(p-1)$ ways. After (s_1) is chosen, (s_2) may be chosen in ${}_{n-p}P_p/p(p-1)$ ways; and then (s_3) may be chosen in ${}_{n-2p}P_p/p(p-1)$ ways; etc. It follows that the cyclic groups, each involving a single cycle of p letters, of which a conjugate of I is the direct product, can be chosen in

$$\frac{{}_nP_p \cdot {}_{n-p}P_p \cdot \cdots {}_{n-p_1p}P_p}{p^{n_1}(p-1)^{n_1}}$$

ways. But these cyclic groups may be interchanged in $n_1!$ ways without altering the conjugate of I under consideration. Hence, observing that, with the notation of § 1, $n - n_1 p = a_0$, it follows that *the number of conjugates of I under G is*

$$\frac{n!}{n_1!\, a_0!\, p^{n_1}(p-1)^{n_1}} .$$

4. *Number of Conjugates of H.* The structure of the normaliser K of I in G, the order of which is $n_1! a_0! p^{n_1}(p-1)^{n_1}$, is now readily determined. The cyclic group generated by s_i is invariant under a metacyclic group M_i, of order $p(p-1)$, which involves only the symbols that s_i affects. Hence K contains the direct product $M = M_1 M_2, \cdots, M_{n_1}$, of order $p^{n_1}(p-1)^{n_1}$. Since s_i and s_j are similar substitutions involving different symbols, we can find a substitution u_{ij} of order 2, which transforms s_i into s_j, and which involves only the symbols affected by s_i and s_j. The substitutions u_{ij} clearly generate a group G_1 which is contained in K and is simply isomorphic with the symmetric group of degree n_1. Since G_1 and M have no substitution in common besides identity, and G_1 transforms M into itself, the order of (M, G_1) is $n_1! p^{n_1}(p-1)^{n_1}$. Now K evidently contains the symmetric group N of order $a_0!$ on the symbols $n_1 + 1, n_1 + 2, \cdots, n$, which I does not

affect. The order of K being equal to the order of (M, G_1, N), it follows that $K = (M, G_1, N)$.

Each Sylow subgroup of G_1, of order p^{m-n_1}, generates with I, a Sylow subgroup of G, of order p^m. Hence the number of Sylow subgroups of G of order p^m equals the number of Sylow subgroups of G_1, of order p^{m-n_1}, multiplied by the number of conjugates of I under G.

Since G_1 is simply isomorphic with the symmetric group of degree n_1, G_1 may be treated as G was. Thus we construct a substitution group I_1, having properties analogous to I, and which has $\dfrac{n_1!}{n_2!\ a_1!\ p^{n_2}(p-1)^{n_2}}$ conjugates under G_1. The substitution group G_2, which bears the same relation to G_1 that G_1 bears to G, is simply isomorphic with the symmetric group of degree n_2, and contains a subgroup I_2, with properties analogous to I, which has $\dfrac{n_2!}{n_3!\ a_2!\ p^{n_3}(p-1)^{n_3}}$ conjugates in G_2.

Continuing in this way we finally arrive at a subgroup I_{k-1} which is a *Sylow* subgroup of the group G_{k-1}, which is simply isomorphic with the symmetric group of degree n_{k-1}. It follows that the number of conjugates of H in G is

$$\frac{n!}{n_1!\ a_0!p^{n_1}(p-1)^{n_1}} \cdot \frac{n_1!}{n_2!\ a_1!\ p^{n_2}(p-1)^{n_2}} \cdots \frac{n_{k-1}!}{n_k!\ a_{k-1}!\ p^{n_k}\ (p-1)^{n_k}}$$
$$= \frac{n!}{a_0!\ a_1! \cdots a_k!\ [p(p-1)]^{n_1+n_2+\cdots+n_k}},$$

since $n_k = a_k$. Observing that $m = n_1 + n_2 + \cdots + n_k$ is the index of the highest power of p that divides $n!$,* we have proved the following

THEOREM. If p is a prime factor of $n!$, and p^m the highest power of p that divides $n!$, and if n is represented in the scale of p in the form $n = a_0 + a_1p + a_2p^2 + \cdots + a_kp^k$, then the symmetric group of degree n contains exactly

$$\frac{n!}{a_0!\ a_1! \cdots a_k!\ p^m(p-1)^m}$$

subgroups of order p^m.

Since each of the a's is less than p, we have the

COROLLARY 1. The order of the normaliser of a Sylow subgroup of order p^m of the symmetric group of arbitrary degree is not divisible by a prime greater than p.

In particular, for $p = 2$, we have the

* Carmichael, *Theory of Numbers*, p. 25.

COROLLARY 2. A Sylow subgroup of even order, of the symmetric group of arbitrary degree, is its own normaliser in this symmetric group.

From Sylow's theorem, we have the

COROLLARY 3. If p is a prime factor of $n!$ and p^m the highest power of p that divides $n!$, and if n is represented in the scale of p in the form $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$, then $\dfrac{n!}{a_0!\ a_1! \cdots a_k!\ p^m} \equiv (-1)^m \bmod p$.

In particular, if $n = p$, we have Wilson's theorem.

5. *Sylow Subgroups of the Alternating Group.* The alternating group of degree n evidently contains as many Sylow subgroups of a given odd order as the symmetric group of degree n. However, it is easy to verify that the alternating groups of the first five degrees do not contain as many Sylow subgroups of even order as the symmetric groups of the same degree.

Let K, of order 2^{m-1} be a Sylow subgroup of the alternating group of degree n, and let H_1 and H_2, of order 2^m, be two Sylow subgroups of the symmetric group of degree n, both of which contain K. By a suitable choice of notation we may suppose that H contains the transpositions (12), (34), (56), $\cdots$, $(r-1, r)$, where r equals n or $n-1$ according as n is even or odd. Then K contains $(12)(34)$, $(12)(56)$, $\cdots$, $(12)(r-1, r)$. Since H_2 leaves at most one symbol fixed, H_2 contains a transposition that involves either 1 or 2, say 1. Let this transposition be $(1i)$. If $i = 2$, $H_1 = H_2$, since each is generated by (12) and K. Otherwise $i > 2$. Now if $n > 5$, K contains the substitution $(12)(x, x+1)$ where neither x nor $x+1$ equals i. Also K contains $(i2)(x, x+1)$, which is the transform of $(12)(x, x+1)$ by $(1i)$. Hence K contains $(12)(x, x+1)(i2)(x, x+1) = (1i2)$, which is of order 3. This being impossible, we have the following

THEOREM. A necessary and sufficient condition that the alternating group of degree n contain as many Sylow subgroups of even order as the symmetric group of degree n is that $n > 5$.

UNIVERSITY OF ROCHESTER.

On Certain Theorems Regarding Summable Series and their Application to the Double and Triple Fourier's Series.*

By Gaylord M. Merriman.

It has been frequently pointed out † that, in many of the applications of ordinary and multiple Fourier's series to physical problems, the theorems regarding their summability afford results of greater generality than those regarding their convergence. Also, in the case of the ordinary Fourier's series, it has been shown ‡ that some of the classical results regarding criteria for its convergence appear as corollaries of Fejér's theorem concerning its summability (C1), if we make use of certain theorems in the general theory of summable series.

The purpose of the present paper is to obtain, in analogous fashion, criteria regarding the convergence of double and triple Fourier's series from known results regarding their summability. In order to do this, it is necessary in the first place to prove two theorems which enable us to infer the convergence of certain types of double and triple series, when we know them to be summable (C1). These theorems are generalizations to the cases of double and triple series of a theorem proved by Pollard in the article cited above. The conditions obtained for the convergence of the double and triple Fourier's series are analogues of those given by Dini for the convergence of the simple series. These conditions are, we believe, new.§

* Presented to the American Mathematical Society, December, 1923 and December, 1924.

† Cf. for example, C. N. Moore, "Applications of the Theory of Summability to Developments in Orthogonal Functions," *Bulletin of the American Mathematical Society*, Vol. 25 (1910), pp. 258-276.

‡ Whittaker and Watson, *Modern Analysis*, ed. 2 (1915), p. 167; Carslaw, *Introduction to the Theory of Fourier's Series*, p. 240; S. Pollard, "On the Deduction of Criteria for the Convergence of Fourier's Series from Fejér's Theorem Concerning their Summability," *Proceedings of the London Mathematical Society*, Series 2, Vol. 15 (1916), pp. 336-339.

§ Conditions for the convergence of the double Fourier's series have been given in form different from ours by Hilda Geiringer, "Trigonometrische Doppelreihen," *Monatshefte für Mathematik und Physik*, Vol. 29 (1918), pp. 65-144, in particular § 41. Our conditions, however, are not of such restrictive nature on the function as are hers. Compare, in this connection, the footnote at the end of the statement of Theorem II.

The conditions for the convergence of the triple Fourier's series are believed to be entirely new. Indeed the whole literature of this part of the subject is quite limited;

§ 1. *A General Theorem Regarding Double Series Summable* (C1).

In the present section, we shall prove a theorem which furnishes a criterion for the convergence of certain double series known to be summable (C1).

Consider first any infinite double series $\sum \mu_{mn}$, which contains an infinity of terms of either sign, and in which

$$(1) \qquad \begin{cases} \lim\limits_{m,n\to\infty} \mu_{mn} = 0, & \\ \lim\limits_{n\to\infty} \mu_{mn} = 0 & \text{(for every } m\text{)}, \\ \lim\limits_{m\to\infty} \mu_{mn} = 0 & \text{(for every } n\text{)}. \end{cases}$$

We shall construct an auxiliary series $\sum a_{mn}$, made up of terms of the μ-series chosen as follows:

Let μ_{MN} be the negative term with greatest modulus in $\sum_1^\infty{}_1^\infty \mu_{mn}$; let $\mu_{M_1N_1}$ be the same in $\sum_{M+1}^\infty{}_{N+1}^\infty \mu_{mn}$; let $\mu_{M_2N_2}$ be the same in $\sum_{M_1+1}^\infty{}_{N_1+1}^\infty \mu_{mn}$, and so on. The M's and N's exist and can be found because of the conditions (1) on the series. Now take

$$a_{mn} = -\mu_{MN} \qquad \left\{ \begin{array}{c} m \leqq M \\ \text{or} \\ n \leqq N,\ m > M \end{array} \right\},$$

$$a_{mn} = -\mu_{M_1N_1} \qquad \left\{ \begin{array}{c} M < m \leqq M_1,\ n > N \\ \text{or} \\ N < n \leqq N_1,\ m > M_1 \end{array} \right\},$$

$$a_{mn} = -\mu_{M_2N_2} \qquad \left\{ \begin{array}{c} M_1 < m \leqq M_2,\ n > N_1 \\ \text{or} \\ N_1 < n \leqq N_2,\ m > M_2 \end{array} \right\},$$

and so on. This gives a sequence that is never increasing with respect to either index, and with respect to both indices, and such that

$$(2) \qquad \mu_{mn} \geqq -a_{mn} \qquad (m = 1, 2, \cdots;\ n = 1, 2, \cdots).$$

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among the few references to be found are the following: W. F. Osgood, "On Cantor's Theorem Concerning the Coefficients of a Convergent Trigonometric Series, with Generalizations," *Transactions of the American Mathematical Society*, Vol. 10 (1909), pp. 337-346; Bess M. Eversull, "On Convergence Factors in Triple Series and the Triple Fourier's Series," *Annals of Mathematics*, Vol. 24 (1922), pp. 141-166; C. C. Camp, "Expansions in Terms of Solutions of Partial Differential Equations: First Paper: Multiple Fourier's Series Expansions," *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 123 ff.; Bess M. Eversull, "The Summability of the Triple Fourier's Series at Points of Discontinuity of the Function Developed," *Transactions of the American Mathematical Society*, Vol. 26 (1924), pp. 313-334.

We need also a second auxiliary series $\sum b_{mn}$, constructed in analogous fashion from the positive terms of the μ-series; we obtain another sequence such that

(2′) $$\mu_{mn} \leqq b_{mn} \qquad (m, n = 1, 2, 3, \cdots).$$

We next establish the following lemma:

LEMMA 1: *If sequences of positive integers $\{p_m\}$ and $\{q_n\}$ can be found such that $p_m \to \infty$ steadily with m, $q_n \to \infty$ steadily with n, while $p_m/m \to 0$ steadily and $q_n/n \to \infty$ steadily; and if*

$$\lim_{m,n\to\infty} p_m q_n a_{mn} = 0,$$

then

$$\lim_{m,n\to\infty} p_m q_n a_{m-p_m,\, n-q_n} = 0.$$

We note first a result derived and used by Pollard,* namely that for sufficiently large values of m and n,

(3) $$p_m \leqq 2p_{m-p_m} \qquad q_n \leqq 2q_{n-q_n}.$$

Hence, multiplying both sides of the product of the inequalities in (3) by $a_{m-p_m,\, n-q_n}$, we find that

$$| p_m q_n a_{m-p_m,\, n-q_n} | \leqq 4 | p_{m-p_m} q_{n-q_n} a_{m-p_m,\, n-q_n} |.$$

Since, by hypothesis, the right-hand side of this inequality approaches zero as m and n become infinite, we have

$$\lim_{m,n\to\infty} p_m q_n a_{m-p_m,\, n-q_n} = 0,$$

and the lemma is proved.

With the aid of the above lemma, we now establish the main theorem of this section:

THEOREM I: *If the series $\sum \mu_{mn}$, satisfying conditions* (1), *is also summable* ($C1$) *to the value s; and if sequences $\{p_m\}$ and $\{q_n\}$ can be found satisfying the conditions of Lemma* 1, *and also such that $p_m q_n b_{mn} \to 0$; and if, furthermore, each of the following expressions*

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* *Loc. cit.*, p. 338.

$$(4)\qquad \left\{ \begin{array}{lll} \dfrac{\sum\limits_{i=m-p_m+1,\ j=n-q_n+1}^{m,\ n} s_{ij}}{p_m q_n} & \dfrac{\sum\limits_{i=m-p_m+1}^{m} s_{in}}{p_m} & \dfrac{\sum\limits_{j=n-q_n+1}^{n} s_{mj}}{q_n} \\ \dfrac{\sum\limits_{i=m+1,\ j=n+1}^{m+p_m,\ n+q_n} s_{ij}}{p_m q_n} & \dfrac{\sum\limits_{i=m+1}^{m+p_m} s_{in}}{p_m} & \dfrac{\sum\limits_{j=n+1}^{n+q_n} s_{mj}}{q_n} \end{array} \right.$$

approaches s as a limit as m and n become infinite; then the series converges to s.

We have

$$s_{mn} = s_{m,n-k} + s_{m-h,n} - s_{m-h,n-k} + \sum_{m-h+1,\ n-k+1}^{m\ n} \mu_{mn};$$

whence, from (2),

$$s_{mn} \geqq s_{m,n-k} + s_{m-h,n} - s_{m-h,n-k} - hk a_{m-h,n-k},$$

or

$$s_{mn} \geqq s_{m,n-k} - s_{m-h,n} - s_{m-h,n-k} - p_m q_n a_{m-p_m,\ n-q_n} \qquad \left\{ \begin{array}{l} 0 \leqq h < p_m \\ 0 \leqq k < q_n \end{array} \right\}.$$

This gives

$$(5)\qquad s_{mn} \geqq \frac{\sum\limits_{j=n-q_n+1}^{n} s_{mj}}{q_n} + \frac{\sum\limits_{i=m-p_m+1}^{m} s_{in}}{p_m} - \frac{\sum\limits_{m-p_m+1,\ n-q_n+1}^{m,\ n} s_{ij}}{p_m q_n} - p_m q_n a_{m-p_m,\ n-q_n}$$

Similarly,

$$(6)\qquad \frac{\sum\limits_{m+1,\ n+1}^{m+p_m,\ n+q_n} s_{ij}}{p_m q_n} \leqq \frac{\sum\limits_{j=n+1}^{n+q_n} s_{mj}}{q_n} + \frac{\sum\limits_{i=m+1}^{m+p_m} s_{in}}{p_m} - s_{mn} + p_m q_n b_{mn}.$$

Lemma 1 and the hypothesis enable us to deduce from (5) that, as m and n become infinite,

$$(7)\qquad \underline{\lim}\ s_{mn} \geqq s;$$

and from (6) that

$$(8)\qquad \overline{\lim}\ s_{mn} \leqq s.$$

A combination of (7) and (8) gives

$$(9)\qquad \lim_{m,n\to\infty} s_{mn} = s,$$

and the theorem is proved.

§ 2. *Application to the Double Fourier's Series.*

In this section, we shall use the general theorem of the preceding section to obtain a criterion for the convergence of the double Fourier's series from known results regarding its summability.

Writing for convenience

$$\phi_1(\alpha,\beta) \equiv f(x+2\alpha, y+2\beta) - f(x+2\alpha, y+0) - f(x+0, y+2\beta) + f(x+0, y+0)$$

$$\psi_1(\alpha) \equiv f(x+2\alpha, y+0) - f(x+0, y+0)$$

$$\psi_2(\beta) \equiv f(x+0, y+2\beta) - f(x+0, y+0)$$

$$\phi_2(\alpha,\beta) \equiv f(x+2\alpha, y+2\beta) - f(x+0, y+0),$$

we are ready to prove the following theorem:

THEOREM II: *If $f(x,y)$ has a Lebesgue integral in the region $(-\pi \leqq x \leqq \pi;\ -\pi \leqq y \leqq \pi)$, and if its development in a double Fourier's series is summable $(C1)$ to the value*

$$(10) \qquad s = \tfrac{1}{4}\{f(x+0, y+0) + f(x+0, y-0) + f(x-0, y+0) + f(x-0, y-0)\}$$

at the point (x,y); then the development also converges to s at that point, provided the integrals of the following types

$$(11) \qquad \int_0^{\epsilon_1}\int_0^{\epsilon_2} \frac{|\phi_1(\alpha\beta)|}{\alpha\beta}\,d\alpha d\beta \qquad \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \frac{|\phi_1(\alpha\beta)|}{\alpha}\,d\alpha d\beta$$

$$\int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} \frac{|\phi_1(\alpha\beta)|}{\beta}\,d\alpha d\beta$$

$$(12) \qquad \int_0^{\epsilon_1} \frac{|\psi_1(\alpha)|}{\alpha}\,d\alpha \qquad \int_0^{\epsilon_2} \frac{|\psi_2(\beta)|}{\beta}\,d\beta$$

*exist, where the ϵ's are arbitrarily small, positive quantities.**

The general term of the double Fourier's series approaches zero as m and n become infinite, by virtue of the generalization to two variables of the Riemann-Lebesgue Theorem.† Hence, sequences $\{p_m\}$ and $\{q_n\}$, satisfying the

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* In the conditions for convergence given by Miss Geiringer (cf. footnote, p. —), the integrands in the last two integrals in (11) are replaced by $|\phi_2(\alpha\beta)|/\alpha\beta$, which, from the form of $\phi_2(\alpha,\beta)$, is seen to restrict the function developed unnecessarily.

The conditions she gives in place of our conditions (11) may be derived by the methods of this article, if we replace (14) by

$$(14') \qquad \frac{1}{pq\pi^2}\Big[\int_0^{\epsilon_1}\int_0^{\epsilon_2}\{\phi_1(\alpha\beta)+\psi_1(\alpha)+\psi_2(\beta)\} + \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2}\phi_2(\alpha\beta) + \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2}\phi_2(\alpha,\beta) + \int_{\epsilon_1}^{\pi/2}\int_0^{\pi/2}\phi_2(\alpha\beta) + \int_0^{\tau/2}\int_0^{\pi/2} f(x+0, y+0)\Big] \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2\alpha} \frac{\sin(2n-q)\beta \sin q\beta}{\sin^2\beta}\,d\alpha d\beta.$$

† Cf. W. H. Young, *Proceedings of the London Mathematical Society*, Vol. 11 (1912), p. 138.

conditions of Theorem I, can be found. To prove our Theorem II, then, it only remains to be shown that the expressions of the form (4) for the double Fourier's series approach s as a limit as m and n become infinite. For this purpose, we shall select the first two as type expressions, and carry through the discussion for them alone, with the remark that the reduction for the others is similar.

We set

$$(13) \qquad \sigma_{mn} \equiv 1/mn \sum s_{ij} = 1/mn\pi^2 \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(x+2\alpha, y+2\beta) \frac{\sin^2 m\alpha}{\sin^2 \alpha} \frac{\sin^2 n\beta}{\sin^2 \beta} \, d\alpha d\beta;$$

and we represent by $\sigma_{mn}^{(i)}$ $(i = 1, 2, 3, 4)$ the expressions analogous to the right-hand member of the above equation, in which the integral is extended only over that portion of the original region of integration which lies in a single quadrant. To each $\sigma_{mn}^{(i)}$ there will correspond an $s_{mn}^{(i)}$ related to it by an equation analogous to (13); and each $\sigma_{mn}^{(i)}$ will approach a limit as m and n become infinite, which limit we shall denote by $s^{(i)}$.

We are now ready to prove two lemmas relating to the behavior of the expressions of the form (4) corresponding to the double Fourier's series of $f(x, y)$.

LEMMA 2: *If $f(x, y)$ satisfies the conditions of Theorem II, the first expression in* (4) *corresponding to its double Fourier's series will approach s as a limit.*

We shall prove that the expression formed by substituting for the s's in (4) the $s_{mn}^{(1)}$'s, will approach $s^{(1)}$ as m and n become infinite. Since the proof for the corresponding expressions with the other upper indices is analogous, our lemma will follow at once.

For the double Fourier's series, we can write this expression as

$$1/pq\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} f(x+2\alpha, y+2\beta) \frac{\sin^2 m\alpha - \sin^2 (m-p)\alpha}{\sin^2 \alpha} \frac{\sin^2 n\beta - \sin^2 (n-q)\beta}{\sin^2 \beta} \, d\alpha d\beta;$$

which reduces to

$$(14) \qquad 1/pq\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \{\phi_1(\alpha\beta) + \psi_1(\alpha) + \psi_2(\beta) + f(x+0, y+0)\} \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2 \alpha} \frac{\sin(2n-q)\beta \sin q\beta}{\sin^2 \beta} \, d\alpha d\beta,$$

each term of which will now be discussed separately.

In order to treat the first term of (14), we break it up into four parts:

$$(15)\quad 1/pq\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} = 1/pq\pi^2 \Big[\int_0^{\epsilon_1}\int_0^{\epsilon_2} + \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} + \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2}\Big],$$

where each integral has the same integrand.

For the first term on the right-hand side of (15), we have

$$\Big|1/\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} \phi_1(\alpha\beta)\ \frac{\sin(2m-p)\alpha \sin p\alpha}{p\sin^2\alpha}\ \frac{\sin(2n-q)\beta \sin q\beta}{q\sin^2\beta}\ d\alpha d\beta\Big|$$

$$\leqq 1/\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} |\phi_1(\alpha\beta)|\ \frac{d\alpha}{\sin\alpha}\ \frac{d\beta}{\sin\beta} < \tfrac{1}{4}\int_0^{\epsilon_1}\int_0^{\epsilon_2} \frac{|\phi_1(\alpha\beta)|}{\alpha\beta}\ d\alpha d\beta$$

since

$$\left|\frac{\sin mx}{\sin x}\right| \leqq m \quad \text{and} \quad \frac{x}{\sin x} \leqq \pi/2 \ \begin{pmatrix} 0 < x \leqq \pi/2 \\ m = 1, 2, \cdots \end{pmatrix}.$$

It follows from the existence of the first integral given in (11) that, given a positive δ, we can choose ϵ_1 and ϵ_2 sufficiently small to make this first term less in absolute value than $\delta/12$.

For the second integral in (15) we have

$$\Big|1/pq\pi^2 \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \phi_1(\alpha\beta)\ \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2\alpha}\ \frac{\sin(2n-q)\beta \sin q\beta}{\sin^2\beta}\ d\alpha d\beta\Big|$$

$$< \frac{1}{4q\sin^2\epsilon_2}\int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \frac{|\phi_1(\alpha\beta)|}{\alpha}\ d\alpha d\beta.$$

From the existence of the second integral in (11), it follows that, since ϵ_1 and ϵ_2 are now fixed, we can choose n sufficiently large to make this term less in absolute value than $\delta/12$. A similar treatment of the third term in (15), and the existence of the third integral in (11), show that it also can, by proper choice of m, be made less in absolute value than $\delta/12$. The last term in (15) can obviously be made less in absolute value than $\delta/12$ for m and n sufficiently large. Therefore, by proper choice of m and n, the first term of (14) can be made less in absolute value than $\delta/3$, contingent on the existence of the integrals in (11).

For the second term in (14), we have *

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* We use the identity

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{\sin^2 \kappa x}{\sin^2 x} = \kappa/2.$$

$$1/pq\pi^2 \int_0^{\pi/2} \int_0^{\pi/2} \psi_1(\alpha) \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2 \alpha} \frac{\sin^2 n\beta - \sin^2(n-q)\beta}{\sin^2 \beta} d\alpha d\beta$$

$$= 1/2p\pi \int_0^{\pi/2} \psi_1(\alpha) \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2 \alpha} d\alpha.$$

We know from Pollard's work * that the existence of the first integral in (12) assures us that we can make this part of (14) less in absolute value than $\delta/3$.

That the third term of (14) can be made less in absolute value than $\delta/3$ follows from a similar reduction and the existence of the second integral in (12).

By an easy reduction, the fourth term of (14) becomes $s^{(1)}$.

Hence we have

$$\left| \frac{\sum_{m-p+1}^{m} \sum_{n-q+1}^{n} s_{ij}^{(1)}}{pq} - s^{(1)} \right| < \delta \qquad \begin{pmatrix} m > H \\ n > K \end{pmatrix}.$$

Similar reductions, as pointed out before, will lead to similar results in the cases of the other $s^{(i)}$, and the lemma is proved.

LEMMA 3: *If the conditions of Lemma 2 are satisfied, then the second of the expressions* (4) *corresponding to the double Fourier's series, approaches s as a limit.*

As in Lemma 2, we carry through the discussion for $s^{(1)}$ only.

For the double Fourier's series, we can write this expression as

$$(16) \qquad 1/p\pi^2 \int_0^{\pi/2} \int_0^{\pi/2} \{\psi_1(\alpha) + \psi_2(\beta) + \phi_1(\alpha, \beta) + f(x+0, y+0)\} \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2 \alpha} \frac{\sin n\beta}{\sin \beta} d\alpha d\beta.$$

For the first term, since

$$1/\pi \int_0^{\pi/2} \frac{\sin \kappa x}{\sin x} dx = \tfrac{1}{2} \dagger,$$

* *Loc. cit.*, p. 339.

† Cf. Bierens de Haan, *Nouvelles Tables d'Intégrales Définies*, p. 115, table 75, No. 7.

we have the following reduction:

$$1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \psi_1(a)\ \frac{\sin(2m-p)a \sin pa}{\sin^2 a}\ \frac{\sin n\beta}{\sin\beta}\ da d\beta.$$
$$= 1/2p\pi \int_0^{\pi/2} \psi_1(a)\ \frac{\sin(2m-p)a \sin pa}{\sin^2 a}\ da.$$

It follows from the discussion of an analogous expression in Pollard's paper, that the existence of the first integral of (12) is a sufficient condition that this term can be made less in absolute value than $\delta/3$, for m sufficiently large.

For the second term of (16), we have

$$\left| 1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \psi_2(\beta)\ \frac{\sin^2 ma - \sin^2(m-p)a}{\sin^2 a}\ \frac{\sin n\beta}{\sin\beta}\ da d\beta \right|$$
$$= \left| 1/2\pi \int_0^{\pi/2} \psi_2(\beta)\ \frac{\sin n\beta}{\sin\beta}\ d\beta \right|$$
$$\leqq \tfrac{1}{4} \int_0^{\pi/2} \left| \frac{\psi_2(\beta)}{\beta} \right| \sin n\beta d\beta.$$

To this we apply the Riemann-Lebesgue Theorem,* which, together with the existence of the second integral in (12), enables us to infer that we can make the expression less in absolute value than $\delta/3$.

The third term of (16) we divide as we did the first term of (14):

$$1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} = 1/p\pi^2 \left\{ \int_0^{\epsilon_1}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2} + \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \right\},$$

each integral having the integrand of (16). The discussions of the first three terms of the right-hand side of the above equation are entirely analogous to those used in the cases of similar terms in Lemma 2. For the last term,

$$\left| 1/p\pi^2 \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \phi_1(a\beta)\ \frac{\sin(2m-p)a \sin pa}{\sin^2 a}\ \frac{\sin n\beta}{\sin\beta}\ da d\beta \right|$$
$$\leqq \frac{1}{4\sin\epsilon_2} \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \left| \frac{\phi_1(a\beta)}{a} \right| \sin(2m-p)a \sin n\beta da d\beta.$$

That this tends to zero, we infer from the generalized Riemann-Lebesgue Theorem and the existence of the second integral in (11). Hence the third term

* Lebesgue, "Sur les Séries Trigonometriques," *Annales Scientifiques de l'Ecole normale supérieure*, Ser. 3, Vol. 20 (1903), pp. 471-473.

of (16) can be made less in absolute value than $\delta/3$, by proper choice of m.

The fourth term reduces immediately into $s^{(1)}$.

Summarizing, we have

$$\lim_{m,n\to\infty} \frac{\sum_{i=m-p+1}^{m} s_{in}{}^{(1)}}{p} = s^{(1)}.$$

Since the discussions for the other $s^{(4)}$ are similar the lemma follows readily.

As we noted before, the other expressions of (4) corresponding to the double Fourier's series can be treated by the methods of either Lemma 3 or Lemma 2. Hence, applications of these lemmas will serve to prove Theorem II, if we remember that the other conditions of Theorem I have been pointed out as fulfilled by the double Fourier's series.

§ 3. *Generalization to Three Variables of Theorem I.*

Consider now any infinite triple series $\sum \mu_{lmn}$, containing an infinity of terms of either sign, and in which

$$(17)\quad \begin{cases} \lim\limits_{lmn\to\infty} \mu_{lmn} = 0 \\ \lim\limits_{l\to\infty} \mu_{lmn} = 0 \text{ (for every } m, n), \qquad \lim\limits_{l,m\to\infty} \mu_{lmn} = 0 \text{ (for every } n), \\ \lim\limits_{m\to\infty} \mu_{lmn} = 0 \text{ (for every } l, n), \qquad \lim\limits_{l,n\to\infty} \mu_{lmn} = 0 \text{ (for every } m), \\ \lim\limits_{n\to\infty} \mu_{lmn} = 0 \text{ (for every } l, m), \qquad \lim\limits_{m,n\to\infty} \mu_{lmn} = 0 \text{ (for every } l). \end{cases}$$

As in § 1, we construct an auxiliary series $\sum a_{lmn}$, such that

$$(18)\qquad \mu_{lmn} \geqq -a_{lmn} \qquad (l, m, n = 1, 2, \cdots).$$

The analogue of Lemma 1 for three variables may be stated as follows:

LEMMA 4: *If sequences of positive integers* $\{p_l\}$, $\{q_m\}$, *and* $\{r_n\}$ *can be found such that* $p_l \to \infty$ *steadily with* l, *while* $p_l/l \to \infty$ *steadily, with similar conditions on the* $\{q_m\}$ *and* $\{r_n\}$, *and if*

$$\lim_{l,m,n\to\infty} p_l q_m r_n a_{lmn} = 0,$$

then

$$\lim_{l,m,n\to\infty} p_l q_m r_n a_{l-p_l,\, m-q_m,\, n-r_n} = 0.$$

The proof of this lemma follows at once along the same lines as that of Lemma 1.

We are now in a position to deduce the analogue of Theorem I:

THEOREM III: *If the series* $\sum \mu_{lmn}$, *satisfying conditions* (17), *is also summable* (*C*1) *to s, and if sequences* $\{p_l\}$, $\{q_m\}$, *and* $\{r_n\}$ *can be found satisfying the conditions of Lemma* 4; *and if also each of the expressions of the types* *

$$(19)\qquad \frac{\sum_{l-p_l+1,\ m-q_m+1,\ n-r_n+1}^{l,m,n} s_{ijk}}{p_l q_m r_n}\ ;\ \frac{\sum_{i=l-p_l+1,\ j=m-q_m+1,}^{l,m} s_{ijn}}{p_l q_m}\ ;\ \frac{\sum_{i=l-p_l+1,}^{l} s_{imn}}{p_l}$$

approaches s as l, m, and n become infinite; then the series converges to s.

By a reduction entirely analogous to that used in § 1 to arrive at equation (5), we have

$$\begin{aligned} s_{lmn} \geqq\ & \frac{\sum_{i=l-p_l+1}^{l} s_{imn}}{p_l} + \frac{\sum_{j=m-q_m+1}^{m} s_{ljn}}{q_m} + \frac{\sum_{k=n-r_n+1}^{n} s_{lmk}}{r_n} \\ & - \frac{\sum_{i=l-p_l+1,\ j=m-q_m+1}^{l,m} s_{jn}}{p_l q_m} - \frac{\sum_{i=l-p_l+1,\ k=n-r_n+1}^{l,n} s_{imk}}{p_l r_n} - \frac{\sum_{j=m-q_m+1,\ k=n-r_n+1}^{m,n} s_{ljk}}{q_m r_n} \\ & + \frac{\sum_{l-p_l+1,\ m-q_m+1,\ n-r_n+1}^{l,m,n} s_{ijk}}{p_l q_m r_n} - p_l q_m r_n\, a_{l-p_l,\ m-q_m,\ n-r_n}\ . \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned} \frac{\sum_{l+1,\ m+1,\ n+1}^{l+p_l,\ m+q_m,\ n+r_n} s_{ijk}}{p_l q_m r_n} \leqq\ & \frac{\sum_{i=l+1,\ j=m+1}^{l+p_l,\ m+q_m,} s_{ijm}}{p_l q_m} + \frac{\sum_{i=l+1,\ k=n+1}^{l+p_l,\ n+r_n} s_{imk}}{p_l r_n} + \frac{\sum_{j=m+1,\ k=n+1}^{m+q_m,\ n+r_n} s_{ljk}}{q_m r_n} \\ & - \frac{\sum_{i=l+1}^{l+p_l} s_{imn}}{p_l} - \frac{\sum_{j=m+1}^{m+q_m} s_{ljn}}{q_m} - \frac{\sum_{k=n+1}^{n+r_n} s_{lmk}}{r_n} \\ & + s_{lmn} - p_l q_m r_n\, a_{lmn}\ . \end{aligned} \tag{21}$$

As l, m, and n become infinite, Lemma 4 and the hypothesis enable us to deduce from (20) that

$$(22)\qquad \underline{\lim}\ s_{lmn} \geqq s;$$

and from (21), that

$$(23)\qquad \overline{\lim}\ s_{lmn} \leqq s.$$

A combination of (22) and (23) gives

* The other eleven expressions of these types appear in the body of the proof of the theorem.

$$\lim_{l,m,n\to\infty} s_{lmn} = s, \tag{24}$$

and the theorem is proved.

§ 4. *Application to the Triple Fourier's Series.*

Theorem III will now be applied to the triple Fourier's series, to deduce a condition for convergence analogous to the Dini conditions mentioned above, and to the generalizations of those conditions obtained in § 2.

We first write for convenience:

$$\begin{aligned}
\phi(\alpha,\beta,\gamma) \equiv{}& f(x+2\alpha, y+2\beta, z+2\gamma) - f(x+2\alpha, y+2\beta, z+0) \\
&- f(x+2\alpha, y+0, z+2\gamma) \\
&- f(x+0, y+2\beta, z+2\gamma) + f(x+2\alpha, y+0, z+0) \\
&+ f(x+0, y+2\beta, z+0) \\
&+ f(x+0, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_1(\alpha,\beta) \equiv{}& f(x+2\alpha, y+2\beta, z+0) - f(x+0, y+0, z+0) \\
\chi_2(\alpha,\gamma) \equiv{}& f(x+2\alpha, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_3(\beta,\gamma) \equiv{}& f(x+0, y+2\beta, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_4(\alpha) \equiv{}& f(x+2\alpha, y+0, z+0) - f(x+0, y+0, z+0) \\
\chi_5(\beta) \equiv{}& f(x+0, y+2\beta, z+0) - f(x+0, y+0, z+0) \\
\chi_6(\gamma) \equiv{}& f(x+0, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\Phi(\alpha,\beta,\gamma) \equiv{}& \phi(\alpha,\beta,\gamma) + \chi_1(\alpha,\beta) + \chi_2(\alpha,\gamma) + \chi_3(\beta,\gamma) - \chi_4(\alpha) \\
&- \chi_5(\beta) - \chi_6(\gamma) + f(x+0, y+0, z+0) \\
=& f(x+2\alpha, y+2\beta, z+2\gamma).
\end{aligned}$$

We are then ready to prove the following theorem:

THEOREM IV: *If $f(x,y,z)$ has a Lebesgue integral in the region $(-\pi \leqq x \leqq \pi;\ -\pi \leqq y \leqq \pi;\ -\pi \leqq z \leqq \pi)$, and if its development in a triple Fourier's series is summable $(C1)$ to the value*

$$s = \tfrac{1}{8}\begin{cases} f(x+0,y+0,z+0) + f(x+0,y+0,z-0) + f(x+0,y-0,z+0) \\ + f(x-0,y+0,z+0) + f(x+0,y-0,z-0) + f(x-0,y+0,z-0) \\ + f(x-0,y-0,z+0) + f(x-0,y-0,z-0) \end{cases} \tag{25}$$

at the point (x,y,z); then the development also converges to s at that point,

provided the integrals of the following types exist:

$$(26)\quad \left\{ \begin{array}{l} \displaystyle\int_0^{\epsilon_1}\int_0^{\epsilon_2}\int_0^{\epsilon_3} \frac{|\phi(\alpha,\beta,\gamma)|}{\alpha\beta\gamma}\,d\alpha d\beta d\gamma \quad \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2}\int_0^{\epsilon_3} \frac{|\phi(\alpha\beta\gamma)|}{\gamma}\,d\alpha d\beta d\gamma \\ \displaystyle\int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2}\int_0^{\epsilon_3} \frac{|\phi(\alpha\beta\gamma)|}{\beta\gamma}\,d\alpha d\beta d\gamma \end{array} \right.$$

$$(27)\quad \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} \frac{|\chi_1(\alpha\beta)|}{\beta}\,d\alpha d\beta \qquad \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \frac{|\chi_1(\alpha\beta)|}{\alpha}\,d\alpha d\beta$$

$$(28)\quad \int_0^{\epsilon_1} \frac{|\chi_4(\alpha)|}{\alpha}\,d\alpha$$

for arbitrary, small, positive ϵ's.

As in § 2, sequences $\{p_l\}$, $\{q_m\}$, and $\{r_n\}$, with the desired properties, can be found, since the general term of the triple Fourier's series approaches zero as l, m, and n become infinite.* Hence, to prove our theorem, it remains only to be shown that the expressions of the form (19) for the triple Fourier's series approach s as a limit as l, m and n become infinite. We shall indicate the discussion for the expressions given explicitly in (19), with the remark that the reductions for the others are similar.

We set

$$(29)\quad \sigma_{lmn} = \frac{1}{lmn\pi^3}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(x+2\alpha, y+2\beta, z+2\gamma)\,\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma;$$

and introduce a notation analogous to that used in § 2: $\sigma_{lmn}^{(i)}$; $s_{lmn}^{(i)}$; $s^{(i)}$; $(i=1, 2, \cdots, 8)$, where $\sigma_{lmn}^{(i)}$ represents the expressions analogous to (29) in which the integration is extended only over that portion of the original region included in a single octant; and where each $s^{(i)}$ is the limit of a corresponding $\sigma_{lmn}^{(i)}$.

We are then ready to prove three lemmas concerning the behavior of the expressions (19) corresponding to the triple Fourier's series.

LEMMA 5: *If $f(x, y, z)$ satisfies the conditions of the above theorem, the first expression in* (19), *corresponding to its triple Fourier's series, will approach s as a limit.*

Proceeding in exactly the same manner as in Lemma 2, we shall establish

* This follows from a further generalization of the Riemann-Lebesgue Theorem.

the result for $s^{(1)}$, and since the proof for the corresponding expressions with the other upper indices is similar, our lemma follows immediately.

For the triple Fourier's series, we can write this expression as

$$\frac{1}{pqr\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} f(x+2\alpha, y+2\beta, z+2\gamma)$$

$$\frac{\sin^2 l\alpha - \sin^2(l-p)\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta - \sin^2(m-q)\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma - \sin^2(n-r)\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma,$$

which reduces to

$$(30)\qquad \frac{1}{pqr\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\Phi(\alpha\beta\gamma)$$

$$\frac{\sin(2l-p)\alpha\sin p\alpha}{\sin^2\alpha}\,\frac{\sin(2m-q)\beta\sin q\beta}{\sin^2\beta}\,\frac{\sin(2n-r)\gamma\sin r\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma;$$

and for the purposes of the lemma, we break it up into eight terms corresponding to the eight different terms of $\Phi(\alpha, \beta, \gamma)$.

We note first that the last of these terms reduces immediately to $s^{(1)}$. The treatment of the first term is an obvious extension of the discussion of the first term in (14); and the others can be reduced immediately to expressions whose treatments are covered by the various discussions in Lemma 2 or in Pollard's paper. Hence it follows from the existence of (26), (27), and (28) that, by proper choice of l, m and n, each of the terms we are considering can be made less in absolute value than $\delta/7$; therefore

$$\lim_{l,m,n\to\infty}\frac{\sum_{l-p+1,\ m-q+1,\ n-r+1}^{l,\ m,\ n} s^{(1)}{}_{ijk}}{pqr} = s^{(1)}.$$

Since similar reductions hold for the other $s^{(i)}$, the lemma follows readily.

LEMMA 6: *If the conditions of Lemma* 5 *are satisfied, then the second of the expressions* (19) *corresponding to the triple Fourier's series will approach s as a limit.*

LEMMA 7: *If the conditions of Lemma* 5 *are satisfied, then the third of the expressions* (19) *corresponding to the triple Fourier's series approaches s as a limit.*

The proofs of these lemmas are entirely analogous to those outlined in

the other lemmas of the paper; these lemmas, then, follow readily, contingent on the existence of the integrals (26), (27), and (28).

The theorem of this section is an immediate consequence of the three lemmas just stated, since it follows from these lemmas that the expressions (19) have the desired properties. Hence, a sufficient condition for the convergence of a summable triple Fourier's series is the existence of the integrals given in (26), (27), and (28), which are the generalizations to three variables of the Dini condition for the convergence of the simple Fourier's series.

UNIVERSITY OF CINCINNATI,
JUNE, 1924.

On the Power Characters of Units in a Cyclotomic Field.

By H. S. Vandiver.

If we consider the relative cyclotomic field $\Omega(K)$, that is, an algebraic field which contains the cyclotomic sub-field $\Omega(a)$, $a = e^{2i\pi/p}$, p an odd prime, then if ω is an integer in $\Omega(K)$ and $\mathfrak{q}$ is an ideal prime, we have

$$\omega^{N(\mathfrak{q})-1} \equiv 1 \pmod{\mathfrak{q}}$$

where $N(\mathfrak{q})$ is the norm of $\mathfrak{q}$ in $\Omega(K)$, and (ω) is prime to $\mathfrak{q}$. Also there is one and but one integer a in the set $0, 1, 2, \cdots, p-1$, such that

$$\omega^{\frac{N(\mathfrak{q})-1}{p}} \equiv a^a \pmod{\mathfrak{q}}$$

provided $\mathfrak{q}$ is prime to p. The expression a^a is known as the power character of ω with respect to $\mathfrak{q}$ in the field $\Omega(K)$. We also write

$$\left\{ \frac{\omega}{\mathfrak{q}} \right\} = a^a.$$

These power characters have been treated by Furtwängler * who gave two laws of reciprocity involving them. Consider the problem of determining $(\omega/\mathfrak{q})$, where ω is a *unit* in $\Omega(K)$. For the special case where $\Omega(K)$ is the field $\Omega(a)$ itself, the power character was determined by Kummer † in terms of known functions of the ideal $\mathfrak{q}$, provided $\Omega(a)$ is a regular field, and ω is any unit of the type

† *Crelle*, 44, 1852, 93-146.

$$e(a) = \sqrt{\frac{(1-a^\gamma)(1-a^{-\gamma})}{(1-a)(1-a^{-1})}}$$

The units $e(a), e(a^\gamma) \cdots, e(a^{\gamma^{\mu-2}})$ (where $\mu = (p-1)/2$, and γ is a primitime root of p) form an independent system, so that any unit in $\Omega(a)$ can be expressed in the form

$$e(a)^{n_0},\ e(a^\gamma)^{n_1},\ \cdots,\ e(a^{\gamma^{\mu-2}})^{n_{\mu-2}}$$

the n's being rational fractions.

* Cf. references to Furtwängler in "The Report on the Theory of Algebraic Numbers," *Bull. Natl. Research Council*, Feb. 1923, p. 20.

In particular if

$$E_n(a) = e(a)\ e(a\gamma)\gamma^{-2n}\ e(a\gamma^2)\gamma^{-4n} \cdots e(a\gamma^{\mu-1})\gamma^{-2(\mu-1)n}$$

then Kummer * proved the relation, where λ is written in place of p, and λ is any odd prime,

$$(1)\quad a \equiv \frac{\gamma^{2n}-1}{2(1+r^{\lambda-2n}-(r+1)^{\lambda-2n})} \cdot \frac{d_0^{\lambda-2n} \log \Psi_r(e^v)}{dv^{\lambda-2n}} \pmod{\lambda}$$

where

$$\left\{\frac{E_n(a)}{\mathfrak{P}}\right\} = a^a,$$

r is any integer $\not\equiv 0$ or -1, modulo λ,

$$\Psi_r(x) = \sum_h x^{-(r+1)h+\operatorname{ind}(g^h+1)}$$

$\mathfrak{P}$ is an ideal prime in $\Omega(a)$ whose norm is q^t, q odd, g is a primitive root of $\mathfrak{P}$, h ranges over the integers 0, 1, 2, $\cdots$, q^t-2, excepting $(q^t-1)/2$, and g is selected so that $g^{(q^t-1)/\lambda} \equiv a \pmod{\mathfrak{P}}$. The symbol

$$\frac{d_0^{\lambda-2n} \log \Psi_r(e^v)}{dv^{\lambda-2n}}$$

means that the $(\lambda-2n)$th derivative of $\log \Psi_r(e^v)$ is taken with respect to v, and $v=0$ substituted in the result. Here e is the Napierian base. Owing to certain applications I expect to make in another paper, it will be necessary to use an extension of Kummer's results to the fields defined by primitive pnth roots of unity, n prime to p, and the present paper will be devoted to the proof of the extended theorem.

I was enabled to carry out this investigation through a grant, relieving me of the duties of teaching from the Heckscher foundation for the Advancement of Research, established by August Heckscher at Cornell University.

1. *Transformations depending on the Jacobi-Kummer cyclotomic function.*

This function was treated by H. H. Mitchell † and is defined as

$$\Psi_{a,b}(\theta) = \sum_h \theta^{-bh+(a+b)\operatorname{ind}(g^h+1)}$$

where θ is now any primitive mth root of unity ($m = pn$) $\operatorname{ind}\omega$ is determined by $g^{\operatorname{ind}\omega} \equiv \omega \pmod{\mathfrak{q}}$, $\mathfrak{q}$ being an ideal prime in $\Omega(\theta)$ whose norm is q^t, g is a primitive root of $\mathfrak{q}$, and h ranges over 0, 1, $\cdots$, q^t-2, with the

* *l. c.*, 106-125.

† *Transactions Amer. Math. Soc.*, 17, 1916, 165-177

exception of $(q^t-1)/2$ if q is odd and zero if $q=2$. Also g is selected so that

$$g^{\frac{q^t-1}{m}} \equiv \theta \pmod{\mathfrak{q}}.$$

As defined here this function is a special case of that treated by Mitchell. It has the properties (Mitchell, *l. c.*, p. 168)

$$\Psi(\theta)\,\Psi(\theta^{-1}) = q^t \tag{2}$$

if $a \not\equiv 0$, $b \not\equiv 0$, $a+b \not\equiv 0 \pmod{m}$, and

$$\Psi_{a,\,0}(\theta) = \Psi_{0,\,b}(\theta) = -1. \tag{3}$$

We now set, if $\beta = e^{2i\pi/n}$

$$D_r = \frac{d^r \log \Psi(e^v\beta)}{dv^r}$$

and

$$D_r(\beta) = \left[\frac{d^r \log \Psi(e^v\beta)}{dv^r}\right]_{v=0}$$

this symbol indicating that $v=0$ is substituted in the function of e^v after differentiation. Put $U = 1/\Psi(e^v\beta)$, we have

$$\frac{d \log \Psi(e^v\beta)}{dv} = U\,\frac{d\,\Psi(e^v\beta)}{dv}$$

and

$$D_r = \frac{d^r\,\Psi(e^v\beta)}{dv^j}\,U + (r-1)\,\frac{d^{r-1}\,\Psi(e^v\beta)}{dv^{j-1}}\cdot\frac{dU}{dv} + \cdots + \frac{d\,\Psi(e^v\beta)}{dv}\cdot\frac{d^{r-1}U}{dv}. \tag{4}$$

Now by (2) the function

$$A = \Psi(x\beta)\,\Psi(x^{-1}\beta^{-1}) - q^t$$

vanishes for $x=a$. Consequently since $(x^p-1)/(x-1)$ is irreducible in the field $\Omega(\beta)$,* we have

$$A = W(x)\,\frac{x^p-1}{x-1}$$

where $W(x)$ is an integral function of x, with coefficients in the field $\Omega(\beta)$. If $V = (e^{vp}-1)/(e^v-1)$ we have

$$\Psi(e^v\beta)\,\Psi(e^{-v}\beta^{-1}) = q^t + VW(e^v) \tag{5}$$

* Weber, *Algebra*, Vol. 1, 2nd ed., 600.

and

$$\Psi(e^{-v}\beta^{-1}) = \frac{q^t + VW}{\Psi(e^v)} = (q^t + VW)U,$$

where $W = W(e^v)$.

We shall now transform this expression by means of variations and extensions of the methods employed by Kummer † for the special case in which θ is a primitive pth root of unity.

We shall use the notation

$$F(e^v\beta)^{(i)} = \frac{d^i F(e^v\beta)}{dv^i}$$

and

$$F(e^v\beta)_0^{(i)} = \left[\frac{d^i F(e^v\beta)}{dv^i}\right]_{v=0},$$

$$(6) \qquad \Psi(e^{-v}\beta^{-1})^{(i)} = U^{(i)}(q^t + VW) + iU^{(i-1)}(VW)^1 + \cdots$$

For $v = 0$ this reduces, since $q^t \equiv 1 \pmod{m}$ to

$$\Psi(e^{-v}\beta^{-1})_0^{(i)} \equiv U_0^{(i)} \pmod{p}$$

$i = 0, 1, 2, \cdots, p-2$.

(4) then gives

$$D_r(\beta) \equiv \Psi(e^v\beta)^r \Psi(e^{-v}\beta^{-1}) + (r-1)\,\Psi(e^v\beta)^{(r-1)}\Psi(e^{-v}\beta^{-1})^{(1)} + \cdots \pmod{p}.$$

If $$\Psi(e^v\beta)_0^{(i)} = f_i(\beta)$$

then $$\Psi(e^{-v}\beta^{-1})_0^{(i)} \equiv (-1)^i f_i(\beta^{-1})$$

and therefore

$$D_r(\beta) \equiv f_r(\beta)f_0(\beta^{-1}) - (r-1)f_{r-1}(\beta)f_1(\beta^{-1}) + \cdots - f_1(\beta)f_{r-1}(\beta^{-1}) \pmod{p}.$$

Now

$$f_i(\beta) = \sum_h (-bh + (a+b)\ \text{ind}\ (g^h+1))^i \beta^{-bh+\text{ind}(g^h+1)}$$

h ranging over the values stated in definition.

Set

$$-bh + (a+b)\ \text{ind}\ (g^h+1) = C_h,$$

$$\text{and } -bk + (a+b)\ \text{ind}\ (g^k+1) = C_k.$$

Then

$$D_r(\beta) \equiv \sum_h \sum_k (C_h^r C_k^0 - (r-1)C_h^{r-1}C_k + \cdots)\beta^{C_h - C_k} \pmod{p},$$

† *Journal für Mathematik*, Vol. 44, p. 122.

or

$$(7) \qquad D_r(\beta) \equiv \sum_h \sum_k C_h (C_h - C_k)^{r-1} \beta^{C_h - C_k} \pmod{p}$$

where h and k each range over the integers $0, 1, \cdots, q^t - 2$, excepting $(q^t - 1)/2$ if q is odd and zero if $q = 2$. Put

$$g^{k-h} \equiv g^{k'} \frac{g^k + 1}{g^h + 1} \equiv g^{k'} \pmod{q}.$$

Then the right hand member of (7) may be written

$$\sum_{k'} \sum_{h'} (-b \text{ in } (g^{h'} - 1) + (a+b) \text{ ind } (g^{k'} - 1) - a \text{ ind } (g^{k'} - g^{h'})) \times (bk' - (a+b)h')^{r-1} \beta^{bk' - (a+b)h'}$$

where h' and k' each range over $1, 2, \cdots, q^t - 2$, but $h' \neq k'$, since in that case ind $(g^{k'} - g^{h'})$ has no meaning. Set, if $pl \equiv -1 \pmod{n}$,

$$(8) \qquad \begin{cases} a = pl \\ b = n - pl. \end{cases}$$

Evidently these values satisfy the restrictions on relation (2). Applying them to the last expression, it reduces modulo p, to

$$n^r \sum_k \sum_h (-\text{ind } (g^{h'} - 1) + \text{ind } (g^{k'} - 1)) (k' - h')^{r-1} \beta^{k'}$$

We re-introduce the values $h' = k'$ since for such values the expression vanishes. We again use the symbols h for h' and k for k'. We shall now employ the generalized function treated by Mitchell.

Let $\theta_1 = e^{2i\pi/n_1 p}$ where n_1 is any divisor of n other than unity. Then using the same Ψ function as before, we have

$$\Psi(\theta_1)\Psi(\theta_1^{-1}) = q^t$$

since (8) gives

$$a \not\equiv 0, \quad b \not\equiv 0, \quad a + b \not\equiv 0 \pmod{n_1 p}.$$

We now go thru exactly the same argument with this relation as with (5). There is obtained

$$D_r(\beta_1) \equiv n^r \sum_k \sum_h (-\text{ind } (g^h - 1) + \text{ind } (g^k - 1)) (k - h)^{r-1} \beta_1^k$$

modulo p, if $\beta_1 = e^{2i\pi/n_1}$. Since n_1 is any divisor of n we can then write

$$D_r(\beta^s) \equiv n^r \sum_k \sum_h (-\text{ind } (g^h - 1) + \text{ind } (g^k - 1)) (k - h)^{r-1} \beta^{ks}$$

$$s = 1, 2, \cdots, n - 1.$$

Since $\sum_{i=1}^{n-1} \beta^{ji} = (\beta^{nj}-1)/(\beta^j-1) - 1 = -1,\ j \not\equiv 0 \pmod{n}$, then

$$(9)\qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s)$$
$$\equiv -n^r \sum_k \sum_h (-\operatorname{ind}(g^h-1) + \operatorname{ind}(g^k-1))(k-h)^{r-1}$$
$$+ n^{r+1} \sum_K \sum_h (-\operatorname{ind}(g^h-1) + \operatorname{ind}(g^K-1))(K-h)^{r-1}$$

c being a particular integer in the set $1, 2, \cdots, n-1$, and K ranges over the integers $< q^t - 1$ which satisfy $K \equiv c \pmod{n}$.

We have

$$\sum_k \sum_h (-\operatorname{ind}(g^h-1) + \operatorname{ind}(g^k-1))(k-h)^{r-1}$$
$$= -\sum_k \sum_h \operatorname{ind}(g^h-1)(k-h)^{r-1} + \sum_k \sum_h \operatorname{ind}(g^k-1)(k-h)^{r-1}$$
$$\equiv \sum_k -(q^t-2)(-h)^{r-1} \operatorname{ind}(g^h-1)$$
$$+ \sum_k (q^t-2)k^{r-1} \operatorname{ind}(g^k-1) \pmod{p},$$

since $\sum_k h^s \equiv 0 \pmod{p}$, $s = 1, 2, \cdots, p-2$.

Noting that $q^t \equiv 1 \pmod{p}$, the above reduces to

$$((-1)^{r-1}-1) \sum_h h^{r-1} \operatorname{ind}(g^h-1) \pmod{p}.$$

Similarly

$$-\sum_K \sum_h \operatorname{ind}(g^h-1)(K-h)^{r-1}$$
$$+ \sum_K \sum_h \operatorname{ind}(g^K-1)(K-h)^{r-1}$$

reduces to

$$-\sum_K K^{r-1} \operatorname{ind}(g^K-1) \pmod{p},$$

if we observe that $\sum_K K^s \equiv 0 \pmod{p}$, $s = 1, 2, \cdots, p-2$, and that there are $(q^t-1)/n$ values of K. Hence (9) becomes, if $r < p$,

$$(10)\qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s)$$
$$\equiv n^r((-1)^r+1) \sum_h h^{r-1} \operatorname{ind}(g^h-1)$$
$$- n^{r+1} \sum_K K^{r-1} \operatorname{ind}(g^K-1) \pmod{p}.$$

Now if

$$\nu = (q^t - 1)/m, \quad \text{then}$$

$$\sum_h h^{r-1} \operatorname{ind} (g^h - 1) \equiv \sum_j \sum_i i^{r-1} \operatorname{ind} (g^{i+jm} - 1) \pmod{p}$$

where $i = 1, 2, \cdots, m-1$, and $j = 0, 1, \cdots, \nu - 1$.
But

$$(g^i - 1)(g^{i+m} - 1) \cdots (g^{i+(\nu-1)m} - 1) \equiv g^{\frac{\nu m(\nu-1)}{2}} (g^{\nu i} - 1) \pmod{q}$$

whence, noting that $\operatorname{ind} (-1) \equiv 0 \pmod{p}$,

$$\sum_j \operatorname{ind} (g^{i+jm} - 1) \equiv \operatorname{ind} (g^{\nu i} - 1) \pmod{p},$$

and since $g^\nu \equiv \theta \pmod{q}$, then

$$\sum_j \operatorname{ind} (g^{i+jm} - 1) \equiv \operatorname{ind} (\theta^i - 1) \pmod{p},$$

and (10) then becomes, for $r < p$,

$$(11) \qquad \sum_{s=1}^{n-1} \beta^{-cs} D_r(\beta^s) \equiv ((-1)^r + 1) \sum_i i^{r-1} \operatorname{ind} (\theta^i - 1) - n^{r+1} \sum_l l^{r-1} \operatorname{ind} (\theta^l - 1) \pmod{p},$$

where $i = 1, 2, \cdots, m-1$, and l ranges over the integers $< m$ such that $l \equiv c \pmod{n}$.

2. If ω is any integer in $\Omega(\theta)$, then by definition

$$\omega^n \equiv g^{n \operatorname{ind} \omega} \pmod{q}$$

$$\omega^{\frac{q^t-1}{m} \cdot n} \equiv g^{\frac{q^t-1}{m} \cdot n \operatorname{ind} \omega} \pmod{q}$$

or, since $g^{(q^t-1)/m} \equiv \theta \pmod{q}$

$$\omega^{\frac{q^t-1}{p}} \equiv \theta^{n \operatorname{ind} \omega} \equiv a^{n \operatorname{ind} \omega} \pmod{q}$$

and

$$\left\{ \frac{\omega}{q} \right\} = a^{n \operatorname{ind} \omega}$$

Writing $I(\omega)$ for $n \operatorname{ind} \omega$ we have from (11),

$$(11a) \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv n^{r-1}((-1)^r + 1) \sum_i i^{r-1} I(\theta^i - 1) - n^r \sum_l l^{r-1} I(\theta^l - 1) \pmod{p}$$

Let $i = ep + d$. If d is fixed, then e ranges over the integers 0, 1, 2, $\cdots$, $n-1$ and then

$$\prod_e (\theta^{ep+d} - 1) = \theta^{nd} \prod_e (\theta^{ep} - \theta^{-d})$$
$$= \theta^{nd} - 1$$

and therefore

$$\sum_i i^{r-1} I(\theta^i - 1) \equiv \sum_d d^{r-1} I(a^{nd} - 1) \pmod{p},$$

$d = 1, 2, \cdots, p-1$; and this may be written

$$\sum_{d=1}^{p-1} d^{r-1} I(a^{nd} - 1) \equiv \sum_{d_1=1}^{(p-1)/2} d_1^{r-1} I(a^{nd_1/2})$$
$$+ \sum_{d_1} I(a^{nd_1/2} - a^{nd_1/2})$$
$$+ \sum_{d_1} (p - d_1)^{r-1} ((I(a^{-nd_1/2}) + I(a^{nd_1/2} - a^{-nd_1/2}))$$

and since the i term exists only in (11a) when r is even, we have

$$\sum_{d=1}^{p-1} d^{r-1} I(a^{nd} - 1) \equiv n I(a) \sum_{d_1=1}^{(p-1)/2} d_1^r \pmod{p}$$

and since r is even then

$$\sum_{d_1} d_1^r \equiv 0 \pmod{p}$$

$r = 2, 4, 6, \cdots, p-3$, and

$$\sum_{d_1} d_1^{p-1} \equiv (p-1)/2 \pmod{p},$$

so that from (11a) we have

$$\sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv -n^r \sum_l l^{r-1} I(\theta^l - 1) \pmod{p}$$

$0 < r < p-1$, and

$$\sum_{s=1}^{n-1} \beta^{-sc} D_{p-1}(\beta^s) \equiv -I(a) - \sum_l l^{p-2} I(\theta^l - 1) \pmod{p},$$

or we may write

$$(12) \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv -n^r \sum_{l=1}^{p-1} I(a^l \beta^c - 1) \pmod{p}$$

$0 < r < p-1$, and

$$(13) \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_{p-1}(\beta^s) \equiv -I(a) - \sum_{l=1}^{p-1} l^{p-2} I(a^l \beta^c - 1) \pmod{p}.$$

These two relations are the extensions of Kummer's relation (1).